# Crossing-Number Critical Graphs have Bounded Path-width


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**Abstract.** The crossing number of a graph $\boldsymbol{G}$, denoted by $\mathrm{cr}(\boldsymbol{G})$, is defined as the smallest possible number of edge-crossings in a drawing of $\boldsymbol{G}$ in the plane. A graph $\boldsymbol{G}$ is crossing-critical if $\mathrm{cr}(\boldsymbol{G}-e) < \mathrm{cr}(\boldsymbol{G})$ for all edges $e$ of $\boldsymbol{G}$. We prove that crossing-critical graphs have "bounded path-width" (by a function of the crossing number), which roughly means that such graphs are made up of small pieces joined in a linear way on small cut-sets. Equivalently, a crossing-critical graph cannot contain a subdivision of a "large" binary tree. This assertion was conjectured earlier by Salazar in [J. Geelen, B. Richter, G. Salazar, *Embedding grids on surfaces*, submitted, 2000].


## 1 Introduction

We begin with the most important definitions here. Additional definitions and comments will be presented in the subsequent section. If $\varrho : [0,1] \to \mathbb{R}^2$ is a simple continuous function, then $\varrho([0,1])$ is a *simple curve*, and $\varrho((0,1))$ is a *simple open curve*.

**Definition.** A graph $\boldsymbol{G}$ is *drawn* in the plane if the vertices of $\boldsymbol{G}$ are distinct points of $\mathbb{R}^2$, and every edge $e = uv \in E(\boldsymbol{G})$ is a simple open curve $\varrho$ such that $\varrho(0) = u$, $\varrho(1) = v$. Moreover, it is required that no edge contains a vertex of $\boldsymbol{G}$, and that no three distinct edges of $\boldsymbol{G}$ share a common point.

We denote by $T(\boldsymbol{G})$ the union of all vertices and all edges of $\boldsymbol{G}$ (viewed as a topological set), and a *face* is a connected component of $\mathbb{R}^2 \setminus T(\boldsymbol{G})$. An *edge crossing* (or a *crossing*) in $\boldsymbol{G}$ is any point of $T(\boldsymbol{G})$ that belongs to two distinct edges. A *drawing* of a graph $\boldsymbol{H}$ is a graph $\boldsymbol{G} \simeq \boldsymbol{H}$ that is drawn in the plane. A graph $\boldsymbol{G}$ is *plane* if $\boldsymbol{G}$ is drawn in the plane without crossings, while $\boldsymbol{G}$ is *planar* if it has a plane drawing.


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We are interested in drawings of (nonplanar) graphs that have a small number of crossings. There are many practical applications of such drawings, including VLSI design [5], and graph visualization [6].

**Definition.** The *crossing number* $\mathrm{cr}(\boldsymbol{H})$ of a graph $\boldsymbol{H}$ is the smallest possible number of edge crossings in a drawing of $\boldsymbol{H}$ in the plane. A graph $\boldsymbol{H}$ is *crossing-critical* if $\mathrm{cr}(\boldsymbol{H}-e) < \mathrm{cr}(\boldsymbol{H})$ for all edges $e \in E(\boldsymbol{H})$. A graph $\boldsymbol{H}$ is *$k$-crossing-critical* if $\boldsymbol{H}$ is crossing-critical and $\mathrm{cr}(\boldsymbol{H}) = k$.

Determining the crossing number of a graph is a hard problem [9] in general, and the crossing number is not even known exactly for complete or complete bipartite graphs. So it is important to study crossing-critical graphs in order to understand what structural properties force the crossing number of a graph to be large. In this work, we prove that if $\boldsymbol{G}$ is a $k$-crossing-critical graph, then $\boldsymbol{G}$ cannot contain a subdivision of a "large in $k$" binary tree. It is known that the latter condition is equivalent to $\boldsymbol{G}$ having "bounded in $k$ path-width", which roughly means that $\boldsymbol{G}$ is made up of small pieces joined in a linear way on small cut-sets. (See formal definitions and statements in the next section.)

**Theorem 1.1.** *There exists a function $f$ such that no $k$-crossing-critical graph contains a subdivision of a (complete) binary tree of height $f(k)$. In particular, $f(k) \leq 6 \cdot (72 \log_2 k + 248) \cdot k^3$.*

## 2 Notation and Comments

We consider finite simple graphs (no loops or multiple edges) in this paper. When reading this paper, it is important to understand the relations and differences between an abstract graph (combinatorial object) and a drawing of a graph (topological object). We mostly speak about actual drawings of graphs. Notice that in our notation an edge as a topological object does not include its endpoints, and a face does not include its boundary. In particular, when we speak about an edge crossing, we do not mean a common end of two edges. We use abstract-graph terms, like a subgraph or a vertex-edge incidence, for graph drawings in their obvious abstract meanings. When we speak about connectivity, we mean, depending on context, either arcwise-connectivity for topological objects, or path-connectivity for graphs.

Further we define the path-width of a graph, and present its basic properties. A notation $\boldsymbol{G} \upharpoonright X$ is used for the subgraph of $\boldsymbol{G}$ induced by the vertex set $X$.

**Definition.** A *path decomposition* of a graph $\boldsymbol{G}$ is a sequence of sets $(W_1, W_2, \ldots, W_p)$ such that $\bigcup_{1\leq i\leq p} W_i = V(\boldsymbol{G})$, $\bigcup_{1\leq i\leq p} E(\boldsymbol{G} \upharpoonright W_i) = E(\boldsymbol{G})$, and $W_i \cap W_k \subseteq W_j$ for all $1 \leq i < j < k \leq p$. The width of a path decomposition is $\max\{|W_i| - 1 : 1 \leq i \leq p\}$. The *path-width* of a graph $\boldsymbol{G}$, denoted by $\mathrm{pw}(\boldsymbol{G})$, is the smallest width of a path decomposition of $\boldsymbol{G}$.

It is known [16] that if $\boldsymbol{G}$ is a minor of $\boldsymbol{H}$, then $\mathrm{pw}(\boldsymbol{G}) \leq \mathrm{pw}(\boldsymbol{H})$. A *binary tree* of height $h$ is a rooted tree $\boldsymbol{T}$ such that the root has degree 2, all other non-leaf vertices of $\boldsymbol{T}$ have degree 3, and every leaf of $\boldsymbol{T}$ has distance $h$ from the root. (A binary tree of height $h$ has $2^{h+1}-1$ vertices.) Since the maximal degree of a binary tree $\boldsymbol{T}$ is 3, a graph $\boldsymbol{H}$ contains $\boldsymbol{T}$ as a minor if and only if $\boldsymbol{H}$ contains $\boldsymbol{T}$ as a subdivision. The important connection between binary trees and path-width was first established by Robertson and Seymour in [16], while the following strengthening is due to Bienstock, Robertson, Seymour and Thomas [4]:

**Theorem 2.1.** (Bienstock, Robertson, Seymour, Thomas)
*(a) If $\boldsymbol{T}$ is a binary tree of height $h$, then* $\mathrm{pw}(\boldsymbol{T}) \geq \frac{h}{2}$.
*(b) If* $\mathrm{pw}(\boldsymbol{G}) \geq p$*, then $\boldsymbol{G}$ contains any tree on $p$ vertices as a minor.*

Notice that the crossing number remains the same if we consider drawings in the sphere instead of the plane, or if we require piecewise-linear drawings. (However, if we require the edges to be straight segments – so called rectilinear crossing number, we get completely different behavior; but we are not dealing with this concept here.) Also, the crossing number is clearly preserved under subdivisions of edges (although not under contractions). Thus it is not an essential restriction when we consider simple graphs only.

One annoying thing about the crossing number is that there exist other possible definitions of it, and we do not know whether they are all equivalent or not. The *pairwise-crossing number* $\mathrm{cr}_{\mathrm{pair}}$ is defined similarly, but it counts the number of crossing pairs of edges, instead of crossing points. The *odd-crossing number* $\mathrm{cr}_{\mathrm{odd}}$ counts the number of pairs of edges that cross an odd number of times only. It clearly follows that $\mathrm{cr}_{\mathrm{odd}}(\boldsymbol{G}) \leq \mathrm{cr}_{\mathrm{pair}}(\boldsymbol{G}) \leq \mathrm{cr}(\boldsymbol{G})$, and it was proved by Tutte [17] that $\mathrm{cr}_{\mathrm{odd}}(\boldsymbol{G}) = 0$ implies $\mathrm{cr}(\boldsymbol{G}) = 0$. The best known general relation between these crossing numbers is due to Pach and Tóth [13] who proved $\mathrm{cr}(\boldsymbol{G}) \leq 2\,\mathrm{cr}_{\mathrm{odd}}(\boldsymbol{G})^2$. Our result is formulated for the ordinary crossing number. However, it holds as well for the pairwise-crossing number as can be checked step by step in the proof.

As noted above, the crossing number is a very difficult graph parameter, both for theoretical study and for practical computations. A lot of work has been done investigating the crossing number of particular graphs, see for example works of Anderson, Richter and Rodney [2, 3], and Richter and Thomassen [14]. For general graphs, reserach so far focused mainly on relations of the crossing number to nonstructural graph properties like the number of edges, for example [1, 12, 13]. On the other hand, crossing-critical graphs play a key role in the investigation of structural properties of the crossing number. Our result gives some insight to the general structure of crossing-critical graphs.

By the Kuratowski theorem, there are only two 1-crossing-critical graphs $\boldsymbol{K}_5$ and $\boldsymbol{K}_{3,3}$, up to subdivisions. On the other hand, an infinite family of 2-crossing-critical graphs with minimum degree at least 3 was found by Kochol in [11]. Moreover, Ding, Oporowski, Thomas and Vertigan [7] have proved that every $\geq$2-crossing-critical graph satisfying certain simple assumptions and having sufficiently many vertices belongs to a well-defined infinite graph class. In particular, these graphs have bounded path-width.

The lastly mentioned result actually speaks about a slightly extended notion of a crossing-critical graph. We say that a graph $\boldsymbol{G}$ is $\geq k$-crossing-critical if $\mathrm{cr}(\boldsymbol{G}) \geq k$ and $\mathrm{cr}(\boldsymbol{G} - e) < k$ for all $e \in E(\boldsymbol{G})$. Richter and Thomassen proved in [15] that if $\boldsymbol{G}$ is a $\geq k$-crossing-critical graph, then $\mathrm{cr}(\boldsymbol{G}) \leq \frac{5}{2}k + 16$ holds. (Note, however, that there exist graphs $\boldsymbol{H}$ and an edge $e \in E(\boldsymbol{H})$, such that $\boldsymbol{H} - e$ is planar but $\mathrm{cr}(\boldsymbol{H})$ is arbitrarily high – such $\boldsymbol{H}$ is not crossing-critical.) Thus we see that, for our main result, there is no significant difference between considering $k$-crossing-critical or $\geq k$-crossing-critical graphs.

Observing the structure of known infinite classes of crossing-critical graphs, Salazar and Thomas formulated the following conjecture, appearing in [8]:

*Conjecture 2.2.* There exists a function $g$ such that any $k$-crossing-critical graph has path-width at most $g(k)$.

The paper [8] proves a weaker statement that the tree-width of a crossing-critical graph is bounded. Our main result, Theorem 1.1, together with Theorem 2.1 immediately imply a solution to Conjecture 2.2:

**Corollary 2.3.** *Let $f$ be the function from Theorem 1.1. If $\boldsymbol{G}$ is a $k$-crossing-critical graph, then the path-width of $\boldsymbol{G}$ is at most $2^{f(k)+1} - 2$.*

## 3 Graph Multicycles

The proof of bounded tree-width for crossing-critical graphs in [8] is based on the following idea: Assuming the contrary, a sequence of disjoint nested cycles that "enclose" all crossed edges is found in the graph, and then the sequence is used to argue that the graph is not crossing-critical. Unfortunately, such an idea does not work directly in our case; but it is still useful to consider certain collections of cycles (instead of single cycles) that "separate" all crossed edges from the rest of the graph.

**Definition.** Let $\boldsymbol{G}$ be a graph drawn in the plane. Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a collection of $m$ distinct (but not necessarily disjoint) cycles in $\boldsymbol{G}$, such that the graph $\boldsymbol{F} = C_1 \cup \ldots \cup C_m \subseteq \boldsymbol{G}$ is plane. A pair $\boldsymbol{M} = (\mathcal{C}, X)$, where $X$ is a face of $\boldsymbol{F}$, is a *multicycle* (in $\boldsymbol{G}$) if the following is satisfied: for all $1 \leq i \leq m$, the cycle $C_i$ is the boundary of some face $Y_i \neq X$ of $\boldsymbol{F}$.

In this situation, the set $X = X(\boldsymbol{M})$ is the *exterior face* of $\boldsymbol{M}$. The faces $Y_i$ bounded by the cycles of $\mathcal{C} = \mathcal{C}(\boldsymbol{M})$ are the *interior faces* of $\boldsymbol{M}$, and their union is denoted by $I(\boldsymbol{M}) = Y_1 \cup \ldots \cup Y_m$. (Notice that a face of $\boldsymbol{F}$ may be neither interior nor exterior.)

For clarity, we will depict a multicycle $\boldsymbol{M}$ with $X(\boldsymbol{M})$ as the unbounded face in the plane. Then, simply speaking, our definition means that the cycles of $\boldsymbol{M}$ are pairwise not "crossed" and not "nested". Figure 1 illustrates the definition. We shall use $\boldsymbol{M}$ to refer to both the multicycle and its underlying graph $\boldsymbol{F} = C_1 \cup \ldots \cup C_m$. This convention allows us to use notations like $V(\boldsymbol{M})$ or $T(\boldsymbol{M})$ in their corresponding graph-meanings.

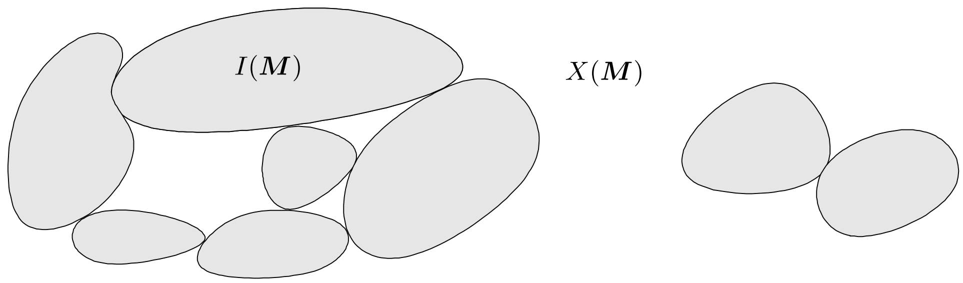


**Fig. 1.** An example of a multicycle $\boldsymbol{M}$ consisting of 8 cycles.

**Definition.** Let $\boldsymbol{M}, \boldsymbol{M}'$ be two multicycles in a graph $\boldsymbol{G}$ such that the union of their underlying graphs is plane. We say that $\boldsymbol{M}$ is *nested* in $\boldsymbol{M}'$, denoted by $\boldsymbol{M} \preceq \boldsymbol{M}'$, if $I(\boldsymbol{M}) \subseteq I(\boldsymbol{M}')$ and $X(\boldsymbol{M}) \supseteq X(\boldsymbol{M}')$. We say that $\boldsymbol{M}$ is *strictly nested* in $\boldsymbol{M}'$, denoted by $\boldsymbol{M} \preccurlyeq\!\!\prec \boldsymbol{M}'$, if $\boldsymbol{M} \preceq \boldsymbol{M}'$ and $|V(\boldsymbol{M}) \cap V(\boldsymbol{M}')| \le 1$.

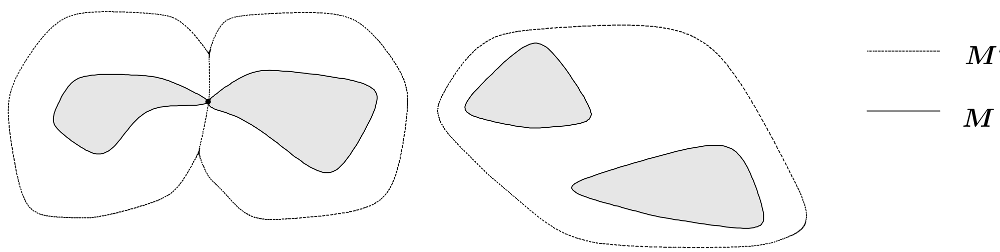


**Fig. 2.** An example of strictly nested multicycles $\boldsymbol{M} \preccurlyeq\!\!\prec \boldsymbol{M}'$ ($\boldsymbol{M}$ consists of 4 cycles, and $\boldsymbol{M}'$ of 3 cycles).

Since a face is a (topologically) connected open set, the definition of $\boldsymbol{M} \preceq \boldsymbol{M}'$ implies that each interior face of $\boldsymbol{M}$ is contained in exactly one interior face of $\boldsymbol{M}'$. It is easy to verify that both relations $\preceq$ and $\preccurlyeq\!\!\prec$ are transitive and antisymmetric, and that $\preceq$ is reflexive. (Focus on inclusions between the interior faces.) An example of nested multicycles is presented in Figure 2.

## 4 Nesting Sequence

In order to motivate the next definitions, we present a short sketch of our proof ideas here. We assume, for a contradiction, that a crossing-critical graph $\boldsymbol{G}$ contains a subdivision of a huge binary tree. (So $\boldsymbol{G}$ is a “bad” graph.) In the first step we denote by $\boldsymbol{M}_0$ the subgraph of $\boldsymbol{G}$ induced by all crossed edges. Then we try to inductively construct a sequence of strictly nested multicycles $\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots$ such that $\boldsymbol{M}_0$ is contained in the interior of $\boldsymbol{M}_1$ and that a large portion of the binary tree still stays in the last exterior face (“outside”). If we find a sufficiently long sequence of these nested multicycles, then we are able to show that $\boldsymbol{G}$ cannot be crossing-critical.

On the other hand, if it is not possible to find the next multicycle for our sequence, then we examine paths connecting the leaves of the "outside tree" to the last multicycle: It is possible that there are not many of these paths, but then again $\boldsymbol{G}$ cannot be crossing-critical. Thus many paths connect leaves of the "outside tree" to the last multicycle, but only a limited number of them may "continue through" our sequence of multicycles all the way to $\boldsymbol{M}_0$. We conclude that a large portion of these paths must "end in one part" of some of the multicycles $\boldsymbol{M}_j$, which again implies a contradiction to $\boldsymbol{G}$ being crossing-critical.

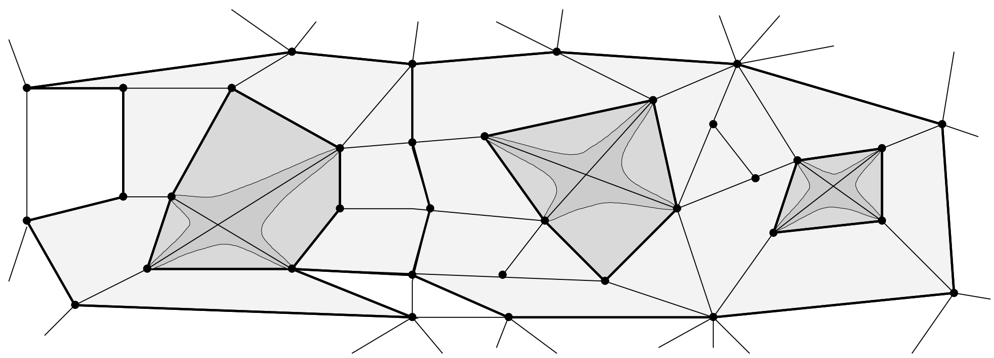

**Fig. 3.** An example of a (strong) 2-nesting sequence in a graph.

The notion of a (strong) nesting sequence is the heart of our proof. However, due to its length and complexity, the definition is divided into two parts, the latter one being presented in Section 6.

**Definition.** Let $\boldsymbol{G}$ be a 2-connected graph drawn in the plane with crossings, and let $c \geq 0$ be an integer. A sequence $\mathcal{M}_c(\boldsymbol{G}) = (\boldsymbol{M}_0, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_c)$ is called a *c-nesting sequence* in $\boldsymbol{G}$ if the following conditions are satisfied:

**(N1)** $\boldsymbol{M}_0$ is the subgraph of $\boldsymbol{G}$ consisting of all crossed edges and their ends. For $1 \leq i \leq c$, $\boldsymbol{M}_i$ is a multicycle in $\boldsymbol{G}$.

**(N2)** Suppose that $c \geq 1$. Then all edges of $\boldsymbol{M}_0$ are contained in $I(\boldsymbol{M}_1)$, and the multicycles are nested as $\boldsymbol{M}_1 \lll \ldots \lll \boldsymbol{M}_c$. For $1 \leq i \leq c$, each interior face of $\boldsymbol{M}_i$ intersects $\boldsymbol{M}_{i-1}$. Moreover, at least one edge of $\boldsymbol{G}$ is in $X(\boldsymbol{M}_c)$.

**(N3)** Suppose that $1 \leq j \leq c$. Let $e = zz'$ be an edge of $\boldsymbol{G}$ such that $z \in V(\boldsymbol{M}_j) \setminus V(\boldsymbol{M}_{j-1})$ and $e \subset I(\boldsymbol{M}_j)$. Then there exists a vertex $t \in V(\boldsymbol{M}_{j-1}) \setminus V(\boldsymbol{M}_j)$ and a path $P$ joining $z$ to $t$ in $\boldsymbol{G}$, such that $e \in E(P)$ and $P$ is internally disjoint from $V(\boldsymbol{M}_{j-1}) \cup V(\boldsymbol{M}_j)$.

We add several comments to this definition. By (N1,2), all edge crossings of $\boldsymbol{G}$ are "enclosed inside" $I(\boldsymbol{M}_1)$, and no multicycle $\boldsymbol{M}_i$, $1 \leq i \leq c$ is involved in a crossing. The purpose of (N3) is to ensure "connectivity between multicycles" in the sequence. An illustration to the definition is in Figure 3. The following is a direct consequence of the above definition

**Lemma 4.1.** *Let* $(\boldsymbol{M}_0, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_c)$, $c \geq 1$ *be a* $c$*-nesting sequence in* $\boldsymbol{G}$*. Then* $|\mathcal{C}(\boldsymbol{M}_c)| \leq |\mathcal{C}(\boldsymbol{M}_{c-1})| \leq \ldots \leq |\mathcal{C}(\boldsymbol{M}_1)|$*. Moreover, if the drawing* $\boldsymbol{G}$ *has* $k$ *crossings, then* $|E(\boldsymbol{M}_0)| \leq 2k$*, and hence* $|\mathcal{C}(\boldsymbol{M}_1)| \leq \frac{1}{2}|E(\boldsymbol{M}_0)| \leq k$*.* ∎

Lastly in this section we show how to find a contradiction if a sufficiently long nesting sequence exists in our crossing-critical graph.

**Lemma 4.2.** *Let* $k$ *be a positive integer. Suppose that there exists a* $(3k-1)$*-nesting sequence in a* 2*-connected graph* $\boldsymbol{H}$ *drawn in the plane. Then* $\boldsymbol{H}$ *is not* $k$*-crossing-critical.*

***Proof.*** Let $\mathcal{M}_{3k-1}(\boldsymbol{H}) = (\boldsymbol{M}_0, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_{3k-1})$ be a $(3k-1)$-nesting sequence in $\boldsymbol{H}$. Informally speaking, our goal is to delete an edge in the exterior face, draw the new graph with fewer crossings, and use pieces of the new drawing to "improve" the drawing $\boldsymbol{H}$. We start with a simple claim.

*Claim 1.* If $2 \leq i \leq 3k-1$ is such that $|\mathcal{C}(\boldsymbol{M}_{i-1})| = |\mathcal{C}(\boldsymbol{M}_i)|$, then, for any interior face $F$ of $\boldsymbol{M}_i$, the subgraph $\boldsymbol{H}_F$ of $\boldsymbol{H}$ induced by the vertices belonging to $F$ is connected.

*Proof.* Let $C'$ be the cycle of $\boldsymbol{M}_i$ bounding $F$. Since $|\mathcal{C}(\boldsymbol{M}_{i-1})| = |\mathcal{C}(\boldsymbol{M}_i)|$, exactly one cycle $C$ of $\boldsymbol{M}_{i-1}$ is contained in the closure of $F$. Moreover, $|V(C) \cap V(C')| \leq 1$, so the subgraph $Q = C - V(C')$ is a cycle or a path. Suppose that there is a component $\boldsymbol{H}_0$ of $\boldsymbol{H}_F$ not containing $Q$. Then $\boldsymbol{H}_0$ is attached to at least two vertices of $C'$ because $\boldsymbol{H}$ is 2-connected, so, in particular, $\boldsymbol{H}_0$ is attached to a vertex $z \in V(C') \setminus V(C)$ by an edge $e = zz'$. Since $e \subset I(\boldsymbol{M}_i)$, by part (N3) of the definition of a nesting sequence, $z'$ is connected by a path in $\boldsymbol{H}_F$ to a vertex of $Q$, a contradiction. □

By the definition of a nesting sequence, there is an edge $e$ of $\boldsymbol{H}$ contained in $X(\boldsymbol{M}_{3k-1})$. If $\boldsymbol{H}$ is $k$-crossing-critical, then there exists a drawing $\boldsymbol{H}^-$ of the graph $\boldsymbol{H} - e$ with fewer than $k$ crossings. We denote by $\boldsymbol{M}_1^-, \boldsymbol{M}_2^-, \ldots, \boldsymbol{M}_{3k-1}^-$ the subgraphs of $\boldsymbol{H}^-$ corresponding to $\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_{3k-1}$ in $\boldsymbol{H}$. These subgraphs are edge-disjoint, so one edge-crossing may involve at most two of them. Thus at least $k$ of $\boldsymbol{M}_2^-, \ldots, \boldsymbol{M}_{3k-1}^-$ are not crossed in $\boldsymbol{H}^-$. Then, by Lemma 4.1, there exists an index $2 \leq i \leq 3k-1$ such that $\boldsymbol{M}_i^-$ is not crossed, and that $|\mathcal{C}(\boldsymbol{M}_{i-1}^-)| = |\mathcal{C}(\boldsymbol{M}_i^-)|$. An illustration to the situation is presented in Figure 4.

Let us denote by $C_1^-, \ldots, C_m^-$ the cycles of $\boldsymbol{M}_i^-$, and by $C_1, \ldots, C_m$ the corresponding cycles of $\boldsymbol{M}_i$. Let $\boldsymbol{H}_j$, $j = 1, \ldots, m$ be the subgraph of $\boldsymbol{H}$ induced by the cycle $C_j$ and its interior, and let $\boldsymbol{H}_j^-$ be the corresponding subgraph of $\boldsymbol{H}^-$. Let $\boldsymbol{H}_0$ be the subgraph of $\boldsymbol{H}$ induced on $\mathbb{R}^2 \setminus I(\boldsymbol{M}_i)$. Notice that $\boldsymbol{H}_0$ is a plane graph. Since the cycle $C_j^-$ is not crossed in $\boldsymbol{H}^-$, the whole graph $\boldsymbol{H}_j^-$ belongs to one region of $T(C_j^-)$ by Claim 1 and the Jordan curve theorem. So there exists a homeomorphic image $\boldsymbol{H}_j^o$ of the graph $\boldsymbol{H}_j^-$ such that the cycle $C_j^-$

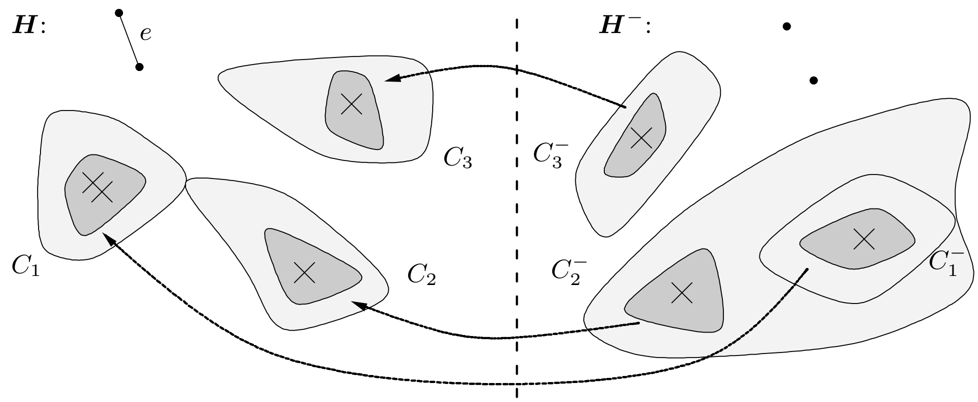


**Fig. 4.** An illustration to how a better drawing of $\boldsymbol{H}$ is obtained using parts of the drawing $\boldsymbol{H}^- \simeq \boldsymbol{H} - e$ that has fewer than $k$ crossings.

becomes $C_j$, and $\boldsymbol{H}_j^o - V(C_j)$ is in the interior face of $C_j$ in $\boldsymbol{H}_0$. Altogether, the graph $\boldsymbol{H}_0 \cup \boldsymbol{H}_1^o \cup \ldots \cup \boldsymbol{H}_m^o$ is isomorphic to $\boldsymbol{H}$, but it contains at most as many crossings as $\boldsymbol{H}^-$, a contradiction. ∎

## 5 Cutting Paths

Unfortunately, it is not always possible to find a sufficiently long nesting sequence in a "bad" graph. In this case we can alternatively exhibit a sequence of "ordered path-cuts" in the graph, as defined next.

**Definition.** Let $\boldsymbol{G}$ be a connected graph drawn in the plane, and let $q \geq 1$ be an integer. A sequence of paths $\mathcal{P}_q(\boldsymbol{G}) = (P_1, P_2, \ldots, P_q)$ in $\boldsymbol{G}$ is called a *q-cutting sequence* if the following conditions are satisfied:

**(C1)** All paths $P_1, P_2, \ldots, P_q$ in $\boldsymbol{G}$ are pairwise disjoint. Each set $V(P_i)$, $1 \leq i \leq q$ is a vertex cut in $\boldsymbol{G}$ (not necessarily minimal).

**(C2)** The set $V(P_1)$ separates the ends of all crossed edges of $\boldsymbol{G}$ from the set $V(P_2) \cup \ldots \cup V(P_q)$.

**(C3)** For $2 \leq i \leq q-1$, the set $V(P_i)$ separates the set $V(P_1) \cup \ldots \cup V(P_{i-1})$ from the set $V(P_{i+1}) \cup \ldots \cup V(P_q)$ in $\boldsymbol{G}$.

We show in this section that, similarly as for nesting sequences, if a sufficiently long cutting sequence exists in a graph, then this graph cannot be crossing-critical. However, notice that a cutting sequence is not a generalization of a nesting sequence, since the paths in a cutting sequence must be vertex-disjoint which is not always true for graph multicycles in a nesting sequence. Moreover, a path is always connected, which need not be (and typically is not) true for a multicycle.

**Lemma 5.1.** *Suppose that there exists a $4k$-cutting sequence in a 2-connected graph $\boldsymbol{H}$ drawn in the plane. Then $\boldsymbol{H}$ cannot be $k$-crossing-critical.*

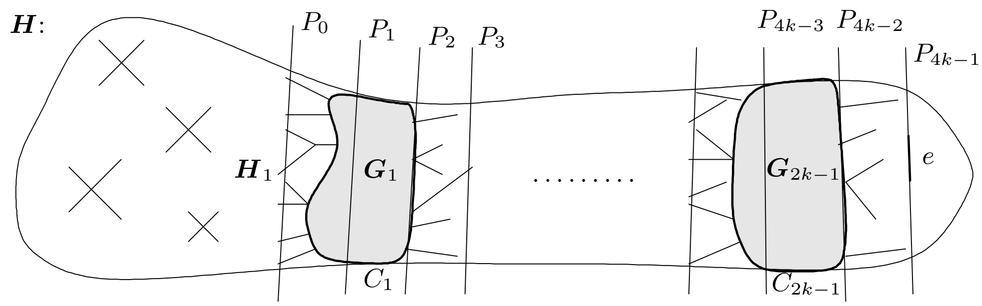


**Fig. 5.** An illustration how a cutting sequence $(P_0, P_1, \ldots, P_{4k-1})$ is used to show that graph $\boldsymbol{H}$ is not $k$-crossing-critical.

***Proof.*** Let the $4k$-cutting sequence in $\boldsymbol{H}$ be $(P_0, P_1, \ldots, P_{4k-1})$. First observe from the definition that the paths $P_0, P_1, \ldots, P_{4k-1}$ of a cutting sequence are "ordered" in the following natural sense: If $Q$ is a path connecting $V(P_0)$ to $V(P_{4k-1})$ in $\boldsymbol{H}$ and $0 < i < j < 4k-1$, then $Q$ first hits $P_i$ before hitting $P_j$. The idea of the proof of this lemma is the same as in Lemma 4.2, but we first need to construct a sequence of disjoint cycles (not nested this time) from successive pairs of paths. An illustration is in Figure 5.

Let $1 \le i \le 2k-1$. We denote by $\boldsymbol{G}'$ the component of $\boldsymbol{H} - V(P_{2i-2})$ containing $P_{2i}$, by $\boldsymbol{G}''$ the component of $\boldsymbol{G}' - V(P_{2i})$ containing $P_{2i+1}$, and by $\boldsymbol{G}_i^\bullet = \boldsymbol{G}' - V(\boldsymbol{G}'')$. Then $\boldsymbol{G}_i^\bullet \supset P_{2i-1} \cup P_{2i}$. (Informally speaking, $\boldsymbol{G}_i^\bullet$ is the subgraph of $\boldsymbol{H}$ "between" $P_{2i-2}$ exclusive and $P_{2i}$ inclusive.) Since $\boldsymbol{H}$ is 2-connected, there are two disjoint paths connecting $V(P_{2i-1})$ to $V(P_{2i})$, and hence some 2-connected component $\boldsymbol{G}_i$ of $\boldsymbol{G}_i^\bullet$ hits both $P_{2i-1}$ and $P_{2i}$. Finally, we denote by $C_i$ the cycle bounding the face $F_i$ of $\boldsymbol{G}_i$ which includes $T(P_{2i-2})$.

*Claim 1.* The graph $\boldsymbol{H} - V(\boldsymbol{G}_i)$ has exactly two components $\boldsymbol{H}_i$, $\boldsymbol{H}_i'$, and $\boldsymbol{H}_i \supseteq P_{2i-2}$, $\boldsymbol{H}_i' \supseteq P_{2i+1}$. All crossed edges of $\boldsymbol{H}$ belong to $\boldsymbol{H}_i$.

*Proof.* The paths $P_{2i-2}$ and $P_{2i+1}$ are in distinct components of $\boldsymbol{H} - V(\boldsymbol{G}_i)$, since any path $Q$ connecting them must intersect both $P_{2i-1}$ and $P_{2i}$, and so the segment of $Q$ between the intersections belongs to $\boldsymbol{G}_i$ (that was chosen as a 2-connected component). Assume that $\boldsymbol{H}''$ is a component of $\boldsymbol{H} - V(\boldsymbol{G}_i)$ containing neither $P_{2i-2}$ nor $P_{2i+1}$. Then $\boldsymbol{H}'' \subset \boldsymbol{G}_i^\bullet$ by the definition of $\boldsymbol{G}_i^\bullet$. Moreover, for similar reasons, since $\boldsymbol{H}$ is 2-connected, there are two disjoint paths in $\boldsymbol{H}$ connecting $V(\boldsymbol{H}'')$ to $V(\boldsymbol{G}_i)$ and contained in $\boldsymbol{G}_i^\bullet$. That contradicts our choice of $\boldsymbol{G}_i$. Finally, any path connecting a crossed edge to $P_{2i}$ must intersect $P_0 \subset \boldsymbol{H}_i$.
□

Clearly, the only vertices of $\boldsymbol{G}_i$ that may be adjacent to $\boldsymbol{H}_i$ are those of $C_i$. The rest of the proof is similar to Lemma 4.2, so we only sketch it. Let $e$ be an edge of $P_{4k-1}$. Suppose that $\boldsymbol{H}$ is $k$-crossing-critical, so there exists a drawing $\boldsymbol{H}^-$ of the graph $\boldsymbol{H} - e$ with fewer than $k$ crossings. We denote by $\boldsymbol{H}_i^-$, $C_i^-$ the subgraphs of $\boldsymbol{H}^-$ corresponding to $\boldsymbol{H}_i$, $C_i$. Then at least one of the $2k-1$ disjoint cycles $C_i^-$, $i \in \{1, \ldots, 2k-1\}$ is not crossed in $\boldsymbol{H}^-$. Since the graph $\boldsymbol{H}_i^-$

is connected, it is contained in one face of $C_i^-$. Hence there is a homeomorphic image of $\boldsymbol{H}_i^-$ that can replace the subgraph $\boldsymbol{H}_i \subseteq \boldsymbol{H}$ in the face $F_i$ without introducing additional crossings. By Claim 1, the new drawing of $\boldsymbol{H}$ has at most as many crossings as $\boldsymbol{H}^-$, a contradiction. ∎

# 6 Strong Nesting Sequence

The definition of a strong nesting sequence is continued here. We advise the reader to compare this definition with the sketch of our proof that was presented in Section 4.

**Definition.** Let $\mathcal{M}_c(\boldsymbol{G}) = (\boldsymbol{M}_0, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_c)$ be a $c$-nesting sequence in a graph $\boldsymbol{G}$. (See on page 6.) Then $\mathcal{M}_c(\boldsymbol{G})$ is called a *strong $c$-nesting sequence* in $\boldsymbol{G}$ if the following is true in addition to conditions (N1–3):

**(N4)** Let $n_0 = 2|E(\boldsymbol{M}_0)|$, and let $V_X(\boldsymbol{M}_c) \subseteq V(\boldsymbol{M}_c)$ denote the set of boundary vertices of the exterior face $X(\boldsymbol{M}_c)$. Suppose that $p \geq 1$ is an arbitrary integer, that $J = \{1, 2, \ldots, \beta\}$ where $\beta = \beta(n_0, c, p) = n_0 p^{2c+1}$, and that $\varphi : J \to V_X(\boldsymbol{M}_c)$ is an arbitrary mapping. In such situation, for some subset $J_0 = \{j_1, j_2, \ldots, j_p\} \subseteq J$, at least one of the cases (a–c) is fulfilled.

(a) It is $c = 0$, and $\varphi(J_0) = \{v\}$ for some vertex $v \in V(\boldsymbol{M}_0)$.

(b) For some $b$, $1 \leq b \leq c$, there exists a vertex $v \in V(\boldsymbol{M}_b)$; and there are $p$ paths $P_i$, $1 \leq i \leq p$ such that $P_i$ connects $\varphi(j_i)$ with $v$ in $\boldsymbol{G}$, all $P_i - v$ are pairwise disjoint, and every $T(P_i)$ is disjoint from $I(\boldsymbol{M}_b) \cup X(\boldsymbol{M}_c)$.

(c) For some $b$, $1 \leq b \leq c$, there exists a path $\boldsymbol{P}$ which is a subpath of a cycle of $\boldsymbol{M}_b$ such that every edge incident in $\boldsymbol{G} - E(\boldsymbol{P})$ with an internal vertex of $\boldsymbol{P}$ is contained in $X(\boldsymbol{M}_b)$; and there are $p$ pairwise disjoint paths $P_i$, $1 \leq i \leq p$ in $\boldsymbol{G}$ such that $P_i$ connects $\varphi(j_i)$ with some $u_i \in V(\boldsymbol{P})$, and every $T(P_i)$ is disjoint from $I(\boldsymbol{M}_b) \cup X(\boldsymbol{M}_c)$.

Again, we add a few comments to this definition. The purpose of (N4) is to "control behavior" of (huge amount of) paths that are coming to $\boldsymbol{M}_c$ from the exterior face. The mapping $\varphi$ represents ends of the incoming paths. Controlling these paths is essential for an inductive construction of a strong nesting sequence later in Lemma 8.1. Notice that some (or even all) of the paths $P_i$ above may have length 0. Actually, the case (a) could be formulated as a special case of (b) for $b = 0$, but we state them separately to avoid unnecessary confusion that may be caused by the fact that $\boldsymbol{M}_0$ is an ordinary subgraph while $\boldsymbol{M}_b$, $b \geq 1$ is a multicycle. Our last comment points out, that, since all crossed edges of $\boldsymbol{G}$ are enclosed in $I(\boldsymbol{M}_1)$, we do not have to bother with crossings when speaking about the paths $P_i$. Figure 3 on page 6 illustrates a strong nesting sequence.

Now we present an obvious statement about a strong 0-nesting sequence.

**Lemma 6.1.** *Let $\boldsymbol{G}$ be a 2-connected graph drawn in the plane with some crossings. Let $\boldsymbol{M}_0$ denote the subgraph of $\boldsymbol{G}$ consisting of all crossed edges. Then $(\boldsymbol{M}_0)$ is a strong 0-nesting sequence in $\boldsymbol{G}$.*

***Proof.*** The condition (N1) is satisfied by the choice of $\boldsymbol{M}_0$, and there is nothing to show in (N2,3). Validity of (N4)(a) follows immediately from $|V(\boldsymbol{M}_0)| \leq 2|E(\boldsymbol{M}_0)| = n_0$ and the pigeon-hole principle. ∎

The next lemma shows how we can iteratively produce a (strong) nesting sequence from nested multicycles in a graph.

**Lemma 6.2.** *Let $\boldsymbol{G}$ be a 2-connected graph drawn in the plane with some crossings. Let $\mathcal{M}_c(\boldsymbol{G}) = (\boldsymbol{M}_0, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_c)$, $c \geq 0$ be a $c$-nesting sequence in $\boldsymbol{G}$. Suppose that $\boldsymbol{N}$ is a multicycle in $\boldsymbol{G}$, that $\boldsymbol{M}_c \lll \boldsymbol{N}$ if $c > 0$ or $I(\boldsymbol{N})$ includes all edges of $\boldsymbol{M}_0$ if $c = 0$, and that $X(\boldsymbol{N})$ contains some edge of $\boldsymbol{G}$. Then there exists a multicycle $\boldsymbol{N}'$ such that $\boldsymbol{N}' \preceq \boldsymbol{N}$, and that $\mathcal{M}_{c+1}(\boldsymbol{G}) = (\boldsymbol{M}_0, \ldots, \boldsymbol{M}_c, \boldsymbol{N}')$ is a $(c+1)$-nesting sequence in $\boldsymbol{G}$. Moreover, $\boldsymbol{N}'$ can be chosen such that, if $\mathcal{M}_c(\boldsymbol{G})$ is a strong nesting sequence, then so is $\mathcal{M}_{c+1}(\boldsymbol{G})$.*

***Proof.*** We define $\mathcal{N}$ to be the collection of all multicycles $\boldsymbol{N}^o$ in $\boldsymbol{G}$ such that $\boldsymbol{N}^o \preceq \boldsymbol{N}$, and $\boldsymbol{M}_c \lll \boldsymbol{N}^o$ if $c > 0$ or $I(\boldsymbol{N}^o)$ includes all edges of $\boldsymbol{M}_0$ if $c = 0$. Since $\boldsymbol{N} \in \mathcal{N}$ and $\mathcal{N}$ is finite, there exists a multicycle $\boldsymbol{N}' \in \mathcal{N}$ that is a minimal element of $\mathcal{N}$ with respect to $\preceq$. The minimality of $\boldsymbol{N}'$ ensures that each interior face of $\boldsymbol{N}'$ intersects $T(\boldsymbol{M}_c)$. We claim that $\boldsymbol{N}'$ satisfies the conclusions of the lemma.

The conditions (N1,2) are clearly true for $\mathcal{M}_{c+1}(\boldsymbol{G})$. We show the validity of (N3): Let $e = zz'$ be an edge of $\boldsymbol{G}$ such that $z \in V(\boldsymbol{N}') \setminus V(\boldsymbol{M}_c)$ and $e \subset I(\boldsymbol{N}')$. We denote by $C'$ the cycle of $\boldsymbol{N}'$ having $e$ in its interior face. Since $\boldsymbol{G}$ is 2-connected, the vertex $z'$ is connected with $C' - z$ by a path $P'$ in $\boldsymbol{G} - z$. Let $P'' \subset P' - z$ be the shortest path connecting $z'$ with a vertex $t \in V(\boldsymbol{M}_c) \cup V(\boldsymbol{N}')$, and let $P = P'' \cup e$. If $t \in V(\boldsymbol{M}_c) \setminus V(\boldsymbol{N}')$, then $P$ is the path required by the condition. Otherwise, $P$ connects two distinct vertices $z, t \in V(C') \subseteq V(\boldsymbol{N}')$, dividing $C'$ into two cycles $C_1', C_2'$. Since $P$ is internally disjoint from $V(\boldsymbol{M}_c)$, the multicycle $\boldsymbol{N}^o \in \mathcal{N}$ obtained from $\boldsymbol{N}'$ by replacing $C'$ with both of $C_1', C_2'$ contradicts the minimality of $\boldsymbol{N}'$.

Now suppose that $\mathcal{M}_c(\boldsymbol{G})$ is a strong nesting sequence. We show the validity of the condition (N4) for $\mathcal{M}_{c+1}(\boldsymbol{G})$: Recall from the definition that $p \geq 1$, that $n_0 = 2|E(\boldsymbol{M}_0)| \geq 4|\mathcal{C}(\boldsymbol{N}')|$ by Lemma 4.1, and that $J = \{1, 2, \ldots, \beta\}$ where $\beta = n_0 p^{2(c+1)+1}$. Assume that $\varphi : J \to V_X(\boldsymbol{N}')$ is a mapping as in (N4). Our idea is to show that either the vertex $v$ or the path $P$ from the conclusions of (N4) can be found right in the multicycle $\boldsymbol{N}'$, or that sufficiently many of the vertices in $\varphi(J)$ can be connected by internally-disjoint paths to vertices of the previous multicycle $\boldsymbol{M}_c$.

If $|\varphi(J)| < \frac{\beta}{p-1}$, then $|\varphi^{-1}(v)| \geq p$ for some vertex $v \in V(\boldsymbol{N}')$, so the part (b) applies for $v$, $b = c + 1$, and $J_0$ being any $p$-element subset of $\varphi^{-1}(v)$. Thus we assume $|\varphi(J)| \geq \frac{\beta}{p-1}$. We denote by $S \subseteq V_X(\boldsymbol{N}')$ the set of all boundary vertices of $X(\boldsymbol{N}')$ that belong to more than one cycle in $\mathcal{C}(\boldsymbol{N}')$. Moreover, we denote by $R$ the set of all vertices $z \in V_X(\boldsymbol{N}')$ for which $z \in V(\boldsymbol{M}_c)$, or for which there exists an edge $e = zz'$ of $\boldsymbol{G}$ contained in $I(\boldsymbol{N}')$. Notice that every cycle in $\mathcal{C}(\boldsymbol{N}')$ intersects $R$ by connectivity. We assign an arbitrary orientation to each cycle in $\mathcal{C}(\boldsymbol{N}')$, and we define a mapping $\vartheta : \varphi(J) \to R \cup S$ as follows: If $x \in \varphi(J) \cap S$, then $\vartheta(x) = x$. If $x \in \varphi(J) \setminus S$, then $\vartheta(x)$ is the point of $R \cup S$ closest to $x$ in the assigned orientation on the cycle $C \in \mathcal{C}(\boldsymbol{N}')$, $x \in V(C)$. Finally, we set $R^o = R \cap \vartheta(\varphi(J))$.

Suppose that $|\vartheta^{-1}(v) \cap V(C)| \geq p$ for some vertex $v \in R^o \cup S$ and some cycle $C \in \mathcal{C}(\boldsymbol{N}')$, $v \in V(C)$. Then by the definition of $\vartheta$, for $U = \vartheta^{-1}(v) \cap V(C)$, there is a path $\boldsymbol{P} \subset C$ on the boundary of $X(\boldsymbol{N}')$ that is internally-disjoint from $R \cup S$, and that $U \subseteq V(\boldsymbol{P})$. It is easy to verify that (N4)(c) is fulfilled for $\boldsymbol{P}$, $b = c + 1$, and for $J_0 \subseteq \varphi^{-1}(U)$ such that $|J_0| = |\varphi(J_0)| = p$. Otherwise, we may assume $|\vartheta^{-1}(v) \cap V(C)| < p$ for all $v$ and $C$ as above. We need the following easy inequality:

*Claim 1.* $\sigma_S = \sum_{x \in S} \left|\{C' \in \mathcal{C}(\boldsymbol{N}') : x \in V(C')\}\right| \leq 4|\mathcal{C}(\boldsymbol{N}')| - 4 \leq n_0.$

*Proof.* It is an easy exercise to show that some cycle $C' \in \mathcal{C}(\boldsymbol{N}')$ intersects the boundary of $X(\boldsymbol{N}')$ in a connected piece. Hence $|S \cap V(C')| \leq 2$ and $C'$ contributes by at most 4 to the sum $\sigma_S$. We finish by induction on the number of cycles in $\boldsymbol{N}'$. □

Using Claim 1 and the previous assumption over all $v$ and $C$, we can estimate $|\varphi(J)| = |\vartheta^{-1}(R^o \cup S)| < p(|R^o| + \sigma_S) \leq p|R^o| + pn_0$, and so $|R^o| > \frac{1}{p}|\varphi(J)| - n_0 \geq \frac{\beta}{p(p-1)} - n_0 \geq n_0\frac{p^{2c+2}-p^{2c+1}}{p-1} = n_0 p^{2c+1}$. It follows from the definition of $R^o$ that there exists a collection of pairwise disjoint paths $Q_z \subset \boldsymbol{N}'$, $z \in R^o$ (possibly of length 0) connecting each vertex of $R^o$ to some vertex in $\varphi(J) \subseteq V(\boldsymbol{N}')$. Moreover, by (N3), for each vertex $x \in R \supseteq R^o$ there exists a path $Q^*_x$ connecting $x$ to a vertex $q^*_x \in V(\boldsymbol{M}_c)$ such that $Q^*_x$ is internally-disjoint from $V(\boldsymbol{M}_c) \cup V(\boldsymbol{N}')$.

If $c = 0$, then, in particular, $|R^o| > n_0 \geq |V(\boldsymbol{M}_0)|$. Hence $q^*_x = q^*_y$ for some distinct $x, y \in R^o$. Then the subpath of $Q^*_x \cup Q^*_y$ connecting $x$ to $y$ divides the cycle $C' \in \mathcal{C}(\boldsymbol{N}')$, $x, y \in V(C')$ into two cycles $C'_1, C'_2$, and so (as in the beginning of this proof) we can form a multicycle $\boldsymbol{N}^o \in \mathcal{N}$, $\boldsymbol{N}^o \preceq \boldsymbol{N}'$ by replacing $C'$ with both of $C'_1, C'_2$, a contradiction to the minimality of $\boldsymbol{N}'$.

If $c \geq 1$, then we define a mapping $\varphi' : R^o \to V(\boldsymbol{M}_c)$ by $\varphi'(x) = q^*_x$. Notice that two paths $Q^*_x, Q^*_y$, $x \neq y \in R^o$ cannot intersect in an internal vertex since that would contradict the minimality of $\boldsymbol{N}'$ similarly as in the previous paragraph. We inductively apply the condition (N4) for $\mathcal{M}_c(\boldsymbol{G})$ and $\varphi'$, obtaining a set $J'_0 = \{r'_1, \ldots, r'_p\} \subset R^o$ and a collection of paths $P'_i$, $1 \leq i \leq p$ connecting vertices $\varphi'(r'_i)$ to a vertex $v'$ or a path $\boldsymbol{P}'$ on $\boldsymbol{M}_b$ (depending on which of (b) or (c) applies). Finally, we define paths $P_i = P'_i \cup Q^*_{r'_i} \cup Q_{r'_i}$, and set $J_0 = \{j_1, \ldots, j_p\}$ such that $\varphi(j_i)$ is the other end of the path $Q_{r'_i}$. It is now routine work to verify that $J_0$ and $P_i$'s satisfy (N4) (b) or (c), respectively. ∎

## 7 Assorted Lemmas

In this section we present several simple lemmas that are used later in the proof.

**Lemma 7.1.** *Let $\boldsymbol{H}$ be a plane graph, and let $\boldsymbol{G}_1, \boldsymbol{G}_2$ be connected subgraphs of $\boldsymbol{H}$. Then either there exists a face in $\boldsymbol{H}$ incident both with a vertex of $\boldsymbol{G}_1$ and a vertex of $\boldsymbol{G}_2$, or there exists a cycle in $\boldsymbol{H}$ disjoint from $\boldsymbol{G}_1 \cup \boldsymbol{G}_2$ and separating $\boldsymbol{G}_1$ from $\boldsymbol{G}_2$.*

***Proof.*** We assume that no face of $\boldsymbol{H}$ is incident both with a vertex of $\boldsymbol{G}_1$ and a vertex of $\boldsymbol{G}_2$. In particular, $V(\boldsymbol{G}_1) \cap V(\boldsymbol{G}_2) = \emptyset$. Let $\boldsymbol{H}' = \boldsymbol{H} - (V(\boldsymbol{G}_1) \cup V(\boldsymbol{G}_2))$. Suppose that $\boldsymbol{G}_1, \boldsymbol{G}_2$ belong to the same face $F'$ of $\boldsymbol{H}'$. This means there exists a sequence of successively adjacent faces $F_1, \dots, F_q$ in $\boldsymbol{H}$ (a "dual path"), such that $F_1$ is incident with $\boldsymbol{G}_1$, $F_q$ is incident with $\boldsymbol{G}_2$, and some edge $e_i$ shared by $F_{i-1}$ and $F_i$, $1 < i \le q$, is not in $E(\boldsymbol{H})$. In particular, $F_1 \cup \ldots \cup F_q \subseteq F'$, and each $e_i$ is incident with $V(\boldsymbol{G}_1) \cup V(\boldsymbol{G}_2)$. However, for some $1 < i < q$, the edge $e_i$ is incident with $V(\boldsymbol{G}_1)$ while the edge $e_{i+1}$ is incident with $V(\boldsymbol{G}_2)$, and hence $F_i$ is incident with both $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, a contradiction. Thus $\boldsymbol{G}_1, \boldsymbol{G}_2$ belong to distinct faces $F_1', F_2'$ of $\boldsymbol{H}'$. The facial walk bounding $F_1'$ then contains a cycle separating $\boldsymbol{G}_1$ from $\boldsymbol{G}_2$. ∎

**Corollary 7.2.** *Let $\boldsymbol{H}$ be a plane graph, let $\boldsymbol{G}$ be a connected subgraph of $\boldsymbol{H}$, and let $F$ be a face of $\boldsymbol{H}$. Then either $F$ is incident with a vertex of $\boldsymbol{G}$, or there exists a cycle in $\boldsymbol{H}$ separating $F$ from $\boldsymbol{G}$.*

***Proof.*** We add a new isolated vertex $w$ into $F$, and we apply the lemma for $\boldsymbol{G}_1 = \{w\}$ and $\boldsymbol{G}_2 = \boldsymbol{G}$. ∎

**Lemma 7.3.** *Let $\boldsymbol{T}$ be a binary tree with root $r$ and height $h \ge 1$, let $q \ge 1$ be an integer, and let $L$ be a subset of $\alpha(h,q) = (2h)^q$ leaves of $\boldsymbol{T}$. Then there exist $q$ pairwise disjoint paths $P_1, \dots, P_q$ in $\boldsymbol{T}$ such that the ends of each $P_i$, $1 \le i \le q$ are in $L$. Moreover, each set $V(P_i)$, $1 < i < q$ is a cut in $\boldsymbol{T}$ separating the set $\{r\} \cup V(P_1) \cup \ldots \cup V(P_{i-1})$ from $V(P_{i+1}) \cup \ldots \cup V(P_q)$.*

***Proof.*** The case of $q = 1$ is trivial, so let $q > 1$. For simplicity we imagine $\boldsymbol{T}$ as a plane tree with the leaves ordered from left to right. We define $P_1$ as the path connecting the left-most with the right-most leaves in $L$. Every component of $\boldsymbol{T} - V(P_1)$ having leaves "between" the ends of $P_1$ is a binary tree again. There are at most $2h-2$ such components, and thus one of the components $\boldsymbol{T}'$ of height $h' < h$ has at least $\frac{(2h)^q-2}{2h-2} \ge (2h)^{q-1} > (2h')^{q-1}$ leaves in $L$. By induction, we find paths $P_2, \dots, P_q$ in $\boldsymbol{T}'$. Notice that the root $r'$ of $\boldsymbol{T}'$ is connected with $r \cup P_1$ by edges in $\boldsymbol{T} - V(\boldsymbol{T}')$. So it remains to verify that $V(P_2)$ separates $\{r\} \cup V(P_1)$ from $V(P_3) \cup \ldots \cup V(P_p)$, which is easy. ∎

## 8 Conclusion of the Proof

We now move towards proving Theorem 1.1. We want to exhibit a contradiction if a $k$-crossing critical graph $\boldsymbol{G}$ contains a subdivision of a sufficiently large binary tree. We consider 2-connected graphs first, since graphs that are not 2-connected can easily be reduced later. Let us fix the value of $k$. Let us denote by $f'(k) = (72 \log_2 k + 248)k^2$, by $f(k) = 6kf'(k)$, and by $f_c(k) = (6k - 2c - 1)f'(k)$.

**Lemma 8.1.** *Let $\boldsymbol{G}$ be a 2-connected $k$-crossing-critical graph that is drawn in the plane with $k$ crossings. Suppose that $\mathcal{M}_c(\boldsymbol{G}) = (\boldsymbol{M}_0, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_c)$, $0 \le c \le 3k-2$ is a strong $c$-nesting sequence in $\boldsymbol{G}$. Moreover, suppose that $\boldsymbol{U} \subseteq \boldsymbol{G}$ is a subdivision of a binary tree of height $f_c(k)$, and that $\boldsymbol{U} \cap \boldsymbol{M}_0 = \emptyset$ if $c = 0$ or $T(\boldsymbol{U}) \subset X(\boldsymbol{M}_c)$ if $c > 0$. Then at least one of the following happens:*

*(a) There exists a multicycle $\boldsymbol{N}$ in $\boldsymbol{G}$ such that $(\boldsymbol{M}_0, \ldots, \boldsymbol{M}_c, \boldsymbol{N})$ is a strong $(c+1)$-nesting sequence in $\boldsymbol{G}$, and that there exists $\boldsymbol{U}' \subseteq \boldsymbol{G}$, $T(\boldsymbol{U}') \subset X(\boldsymbol{N})$ which is a subdivision of a binary tree of height $f_{c+1}(k)$.*

*(b) There exist $3k-1$ multicycles $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_{3k-1}$ in $\boldsymbol{G}$ such that $(\boldsymbol{M}_0, \boldsymbol{N}_1, \ldots, \boldsymbol{N}_{3k-1})$ is a $(3k-1)$-nesting sequence in $\boldsymbol{G}$.*

*(c) There exist $4k$ paths that form a $4k$-cutting sequence in $\boldsymbol{G}$.*

***Proof.*** To make our arguments as smooth as possible, we start with several useful conventions: Recall that while $\boldsymbol{M}_i$, $1 \le i \le c$ are multicycles, $\boldsymbol{M}_0$ is an ordinary subgraph of $\boldsymbol{G}$. However, as this proof speaks only about what happens "outside of $\boldsymbol{M}_c$", we do not want to formally distinguish between $\boldsymbol{M}_0$ and $\boldsymbol{M}_i$. So for now we define $X(\boldsymbol{M}_0)$ to be the face of $\boldsymbol{M}_0$ containing $\boldsymbol{U}$, and $I(\boldsymbol{M}_0)$ to be the union of all edges of $\boldsymbol{M}_0$ (not including the vertices).

All trees we consider in this proof are plane and rooted, with the root on top and the branches growing down. The leaves are naturally ordered from left to right by this drawing. (Notice that such a view "ties" the graph $\boldsymbol{G}$ to the plane – we may no longer treat the unbounded face as equivalent to bounded faces.) Suppose that $\boldsymbol{T}$ is a binary tree, and $\boldsymbol{T}'$ is a subdivision of $\boldsymbol{T}$. A *node* of $\boldsymbol{T}'$ is a vertex of $\boldsymbol{T}'$ that is also a vertex of $\boldsymbol{T}$. We say that a node $u$ of $\boldsymbol{T}'$ is at *level* $l \ge 0$ if $u$ has in $\boldsymbol{T}$ distance $l$ from the root. If $u$ is a node of $\boldsymbol{T}'$, and $e$ is the first edge of the path connecting $u$ with the root of $\boldsymbol{T}'$, then $\boldsymbol{T}'(u)$ denotes the component of $\boldsymbol{T}' - e$ including $u$, and $\boldsymbol{T}'(u; l)$ denotes the subtree induced by the first $l$ levels of $\boldsymbol{T}'(u)$.

Due to the length and complexity of this proof, we present an informal description of our ideas first:

- We divide the "tree of height $f_c(k)$" in $X(\boldsymbol{M}_c)$ into layers of heights $f'(k)$, $f'(k)$, and $f_{c+1}(k)$. We try to "isolate" leaves of some middle-layer subtree from the rest of the graph. If we succeed, we either get (a), or we use Lemma 7.3 to get (c).
- If we are not successful in the previous step, then, using Lemma 7.1, we argue that most of the middle-layer subtrees are "cut in half" by a closed curve in $T(\boldsymbol{G})$. If sufficiently many such curves do not intersect $\boldsymbol{M}_c$, then they are graph cycles in $\boldsymbol{G}$, and we use them to construct a multicycle for (a).
- Otherwise, most of middle-layer subtrees are connected by pairwise internally-disjoint paths to vertices of $\boldsymbol{M}_c$. In such case we apply the property (N4) from the definition of a strong nesting sequence for the ends of these paths, and Lemma 7.3 for the top-layer subtree, in order to obtain (b) or (c).

Recall that $\boldsymbol{U} \subset \boldsymbol{G}$ is the subdivision of a binary tree of height $f_c(k)$ contained in $X(\boldsymbol{M}_c)$. It is important to keep in mind that no vertex of $\boldsymbol{U}$ is incident with a crossed edge of $\boldsymbol{G}$. By a direct application of Lemma 4.1, the set $T(\boldsymbol{M}_c)$ (or

$I(\boldsymbol{M}_c)$) has at most $k$ connected components, and this fact is used frequently in the proof. Suppose that $u$ is a node of $\boldsymbol{U}$ at level $2f'(k)$. Then $\boldsymbol{U}(u)$ is a subdivision of a binary tree of height $f_c(k) - 2f'(k) = f_{c+1}(k)$. If $\boldsymbol{L}$ is a multicycle such that $T(\boldsymbol{M}_c) \subset I(\boldsymbol{L})$ and $T(\boldsymbol{U}(u)) \subset X(\boldsymbol{L})$, then we may use Lemma 6.2 and conclude that (a) happens. In this situation we call $\boldsymbol{L}$ a *good multicycle* in $\boldsymbol{G}$.

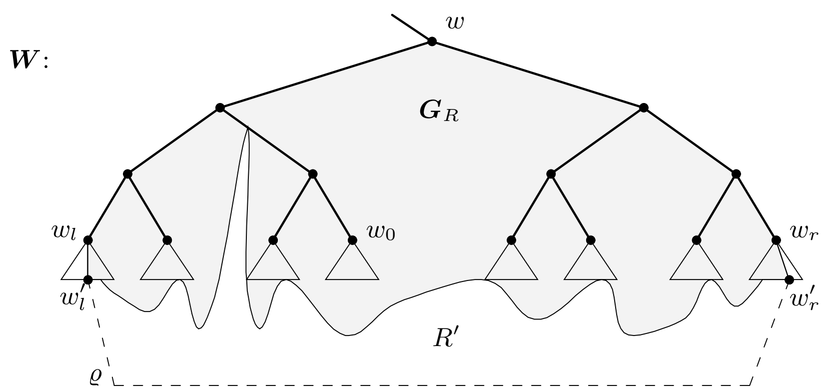


**Fig. 6.** An illustration to the situation in Claim 1.

Let $w$ be a node of $\boldsymbol{U}$ at level $f'(k)$, and let $\boldsymbol{W} = \boldsymbol{U}(w; f'(k))$. We denote by $w_l$, $w_r$ the left-most and right-most, respectively, leaves of $\boldsymbol{W}$. Suppose that there exists a face $R'$ of $\boldsymbol{G}$ which is incident both with a vertex $w'_l$ of $\boldsymbol{U}(w_l)$ and a vertex $w'_r$ of $\boldsymbol{U}(w_r)$. Then there exists a curve $\varrho$ connecting $w'_l$ with $w'_r$ inside $R'$, and a path $P$ connecting $w'_l$ with $w'_r$ in $\boldsymbol{W} \cup \boldsymbol{U}(w_l) \cup \boldsymbol{U}(w_r)$. By the Jordan curve theorem, the simple closed curve $\varrho \cup T(P)$ divides the plane into two regions, exactly one of which, say $R^o$, contains $T(\boldsymbol{W}) \setminus T(P)$. Set $R = R^o \cup \varrho \cup T(P)$, so $T(\boldsymbol{W}) \subset R$. Notice that if $T(\boldsymbol{M}_c)$ intersects $R$, then some component of $T(\boldsymbol{M}_c)$ is a subset of $R \setminus R'$. Since no region $R_1$, defined in a corresponding way for another node $w_1 \neq w$ at level $f'(k)$, can intersect $R \setminus R'$, at most $k$ such regions like $R$ may intersect $T(\boldsymbol{M}_c)$. So suppose for now that does not happen. (See an illustration in Figure 6.)

*Claim 1.* If, for $w, \boldsymbol{W}, R', R$ chosen as above, $R \subseteq X(\boldsymbol{M}_c)$ holds, then one of (a) and (c) holds.

*Proof.* Let $w_0$ be a leaf of $\boldsymbol{W}$ other than $w_l, w_r$. Then since $w_0 \notin V(P)$, the whole subtree $\boldsymbol{U}(w_0)$ is in $R$. Let $\boldsymbol{G}_R$ be the plane subgraph of $\boldsymbol{G}$ contained in $R$. By Corollary 7.2, either the face of $\boldsymbol{G}_R$ containing $R'$ is incident with a vertex of $\boldsymbol{U}(w_0)$, or there is a cycle $C \subset \boldsymbol{G}_R$ separating $\boldsymbol{U}(w_0)$. If the latter happens for some $w_0$, then $C$ forms a good multicycle (with $\boldsymbol{U}(w_0)$ in its exterior), so we are done by (a). Otherwise, returning back to $\boldsymbol{G}$, the face $R'$ is incident with some vertex of each tree $\boldsymbol{U}(w_0)$ for $w_0$ ranging over all leaves of $\boldsymbol{W}$.

We claim that in this situation (c) applies: If $P_1$ is an arbitrary path in $\boldsymbol{W}$ connecting two of its leaves $w_1, w_2$, then $P_1$ can be "prolonged" into a path $P_1^+ \supseteq P_1$, $P_1^+ \subset \boldsymbol{W} \cup \boldsymbol{U}(w_1) \cup \boldsymbol{U}(w_2)$ such that both ends of $P_1^+$ are incident with the face $R'$ in $\boldsymbol{G}$. Notice that $\boldsymbol{W}$ has height $f'(k)$ and $2^{f'(k)}$ leaves, and that $2^{f'(k)} = 2^{(72 \log_2 k + 248)k^2} = (2^{62}k^{18})^{4k^2} > (2f'(k))^{4k}$. Therefore we may apply Lemma 7.3, obtaining a sequence of $4k$ paths $P_1, \ldots, P_{4k}$ as described by the lemma. It is easy to verify, using the fact that $\boldsymbol{G}_R$ is plane, that $P_1^+, \ldots, P_{4k}^+$ is a $4k$-cutting sequence in $\boldsymbol{G}$. □

We define a graph $\boldsymbol{G}^\bullet$ as the plane graph obtained from $\boldsymbol{G}$ by adding, for every crossing $x$ of edges $e, e'$, a new vertex subdividing both $e, e'$ in the point $x$. Notice that $\boldsymbol{G}$, $\boldsymbol{G}^\bullet$ have the same collection of faces. If Claim 1 does not apply, then for at least $2^{f'(k)} - k$ nodes $w \in V(\boldsymbol{U})$ at level $f'(k)$ and for $w_l, w_r$ defined as above, there is no face of $\boldsymbol{G}$ incident both with a vertex of $\boldsymbol{U}(w_l)$ and a vertex of $\boldsymbol{U}(w_r)$. Thus, by Lemma 7.1, the plane graph $\boldsymbol{G}^\bullet$ contains a cycle $C_w$ separating $\boldsymbol{U}(w_l)$ from $\boldsymbol{U}(w_r)$. Without loss of generality we may assume that $C_w$ is a union of a nonempty path $P'_w \subset \boldsymbol{U}$ (possibly being just one vertex), and of a path or cycle $P_w$ which is disjoint from $\boldsymbol{U} - V(P'_w)$. We assign an orientation to $T(C_w)$ such that $w_l$ belongs to the right-hand region of $T(C_w)$. If $T(C_w) \subset X(\boldsymbol{M}_c)$, then $C_w$ is also a cycle of $\boldsymbol{G}$.

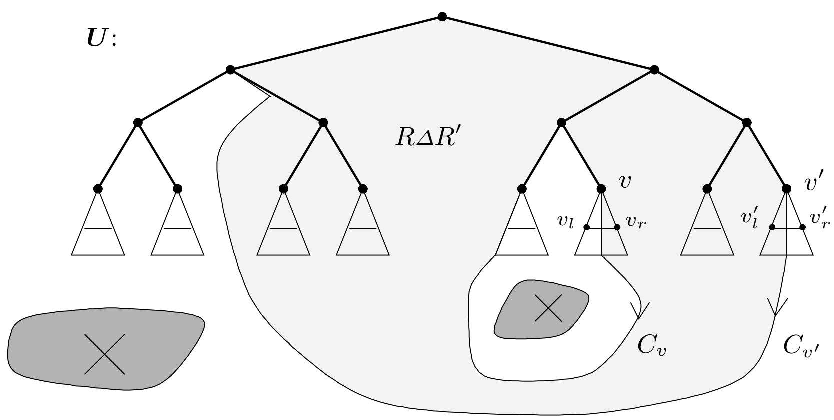


**Fig. 7.** An illustration to the situation in Claim 2.

*Claim 2.* Suppose that, for at least $2^{k+2}$ nodes $w \in V_0 \subseteq V(\boldsymbol{U})$ at level $f'(k)$, the whole cycle $C_w$ is in $X(\boldsymbol{M}_c)$. Then (a) holds.

*Proof.* There exist at most $2^k < \frac{1}{3}|V_0|$ distinct collections of components of $I(\boldsymbol{M}_c)$. Therefore there are three distinct nodes $v, v', v'' \in V_0$ such that the right-hand regions $R, R', R''$ of the oriented cycles $T(C_v), T(C_{v'}), T(C_{v''})$, respectively, share the same collection of components of $I(\boldsymbol{M}_c)$. We denote by $v_l, v_r$

and $v_l', v_r'$ the left-most and right-most leaves of $\boldsymbol{U}(v; f'(k))$ and of $\boldsymbol{U}(v', f'(k))$. See Figure 7.

Since any cycle $C_w$, $w \in V_0$ may intersect at most one subtree $\boldsymbol{U}(w')$, where $w'$ ranges over the nodes of $\boldsymbol{U}$ at level $f'(k)$ other than $w$; we may assume, after possible renaming, that the cycle $C_{v'}$ does not intersect $\boldsymbol{U}(v)$. (The node $v''$ was needed only to perform this renaming.) That means either both $\boldsymbol{U}(v_l), \boldsymbol{U}(v_r)$ are in $R'$, or both $\boldsymbol{U}(v_l), \boldsymbol{U}(v_r)$ are disjoint with $R'$. Thus one of $\boldsymbol{U}(v_l), \boldsymbol{U}(v_r)$, say $\boldsymbol{U}(v_r)$, belongs to the symmetric difference $S = R \Delta R'$. Notice that $S \subset X(\boldsymbol{M}_c)$. If $|V(C_v) \cap V(C_{v'})| \leq 1$, then $(\{C_v, C_{v'}\}, S)$ clearly is a good multicycle in $\boldsymbol{G}$. Otherwise, the graph $C_v \cup C_{v'}$ is 2-connected, so it contains a cycle $C_0$ bounding a face $S_0 \subseteq S$, $T(\boldsymbol{U}(v_r)) \subset S_0$, and hence $(\{C_0\}, S_0)$ is a good multicycle again. Thus (a) follows. $\square$

Finally, we focus on the case that neither Claim 1, nor Claim 2 may be applied. That means, for at least $2^{f'(k)} - k - 2^{k+2} \geq 2^{f'(k)-1}$ nodes $w$ of $\boldsymbol{U}$ at level $f'(k)$, there is a cycle $C_w$ in $\boldsymbol{G}^\bullet$ which separates $\boldsymbol{U}(w_l)$ from $\boldsymbol{U}(w_r)$, and which intersects $T(\boldsymbol{M}_c)$. Moreover, we may assume that for none of these nodes $w$ there is such a separating cycle $C_w'$ not intersecting $T(\boldsymbol{M}_c)$. In this situation we find a path $Q_w \subset C_w$ that connects $w$ with some vertex in $T(\boldsymbol{M}_c)$, and that $Q_w$ is internally disjoint both from $T(\boldsymbol{M}_c)$ and from $\big(\boldsymbol{U} - V(\boldsymbol{U}(w))\big) \cup \boldsymbol{U}(w_l) \cup \boldsymbol{U}(w_r)$. We say that $Q_w$ is a *good connection* from $w$ to $\boldsymbol{M}_c$. Then two good connections $Q_w, Q_{w'}$, $w \neq w'$ do not intersect except in $V(\boldsymbol{M}_c)$ by the previous assumption.

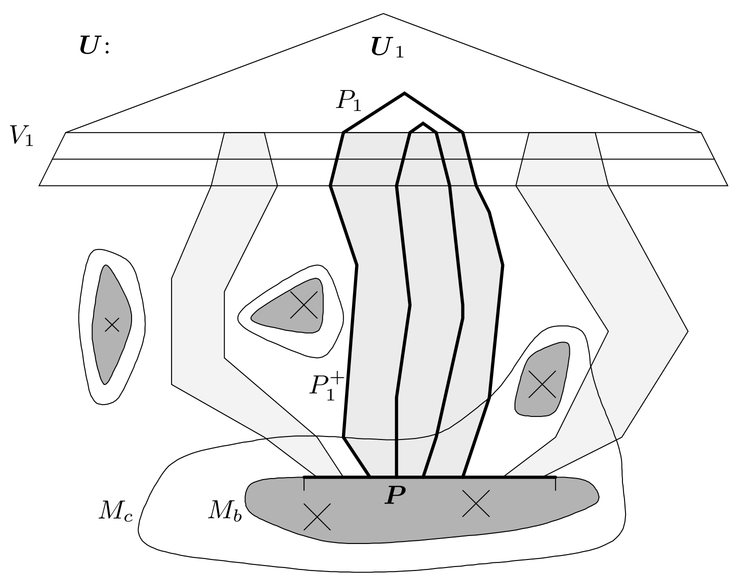


**Fig. 8.** An illustration to the situation in Claim 3.

*Claim 3.* Suppose that, for at least $2^{f'(k)-1}$ nodes $w \in V_1 \subset V(\boldsymbol{U})$ at level $f'(k)$, there is a good connection $Q_w$ from $w$ to $\boldsymbol{M}_c$. Moreover, suppose that all the paths $Q_w$, $w \in V_1$ are pairwise internally-disjoint. Then one of (b) and (c) holds.

*Proof.* Let $V_1 = \{v_1, v_2, \ldots, v_m\}$ for $m \geq 2^{f'(k)-1}$, and let $J = \{1, \ldots, m\}$. We define a mapping $\varphi : J \to V(\boldsymbol{M}_c)$ by the following rule: The image $\varphi(i)$ is the vertex of $V(Q_{v_i}) \cap V(\boldsymbol{M}_c)$. (Clearly $\varphi(i)$ lies on the boundary of $X(\boldsymbol{M}_c)$.) We set $p = (k+1)\alpha(f'(k), 4k)$ where $\alpha(h, p) = (2h)^p$ is the bound from Lemma 7.3. We are going to apply condition (N4) of the definition of a strong nesting sequence onto $\varphi$ and $p$. To do that we first need to verify $m \geq \beta(n_0, c, p)$ where $n_0 \leq 4k$ and $c \leq 3k - 2$:

$$\log_2 \beta\left(4k, 3k-2, (k+1)\alpha(f'(k), 4k)\right) = \log_2\left(4k\left((k+1)(2f'(k))^{4k}\right)^{2(3k-2)+1}\right) \leq$$

$$\leq 2+6k\log_2(k+1)+24k^2(1+\log_2 f'(k)) \leq 8k^2+24k^2\left(1+\log_2((72\log_2 k+248)k^2)\right) \leq$$

$$\leq 8k^2 + 24k^2\,(1 + 3\log_2 k + 9) - 1 = (72\log_2 k + 248)k^2 - 1 = f'(k) - 1$$

Consider first the case when (N4)(c) happens. (The case is illustrated in Figure 8.) Then there exists a subset $V_1' \subset V_1$, $|V_1'| = p$ such that the ends of $Q_w$, $w \in V_1'$ other than $w$ form the set $\varphi(J_0)$ as given by (N4). If $\varphi(j_i) \in V(Q_w)$, then the path $Q_w^+ = Q_w \cup P_i$ connects $w$ with a vertex of the path $\boldsymbol{P} \subset \boldsymbol{M}_b$ from (N4)(c). Moreover, the paths $Q_w^+$, $w \in V_1'$ are pairwise disjoint. Let $\boldsymbol{U}_1 = \boldsymbol{U}(r; f'(k))$ where $r$ is the root of $\boldsymbol{U}$. Consider now the plane subgraph $\boldsymbol{G}_Q = \boldsymbol{U}_1 \cup \boldsymbol{P} \cup \left(\bigcup_{w\in V_1'} Q_w^+\right)$ which has $p$ faces. At most $k$ faces of $\boldsymbol{G}_Q$ may contain components of $I(\boldsymbol{M}_b)$, and at most one other may be the unbounded face. Thus there is a set $V_1'' \subset V_1'$, $|V_1''| = p' \geq \frac{p}{k+1}$, such that one can write $V_1'' = \{v_1', \ldots, v_{p'}'\}$; and for all $i = 1, \ldots, p'-1$, the paths $Q_{v_i'}^+$, $Q_{v_{i+1}'}^+$ share the boundary of one bounded face of $\boldsymbol{G}_Q$ disjoint from $I(M_b)$.

We apply Lemma 7.3 for the tree $\boldsymbol{U}_1$ and the set $L = V_1''$ of leaves of $\boldsymbol{U}_1$. Since $|V_1''| \geq \frac{p}{k+1} = \alpha(f'(k), 4k)$, we get a sequence of $4k$ disjoint paths $P_1', \ldots, P_{4k}'$ in $\boldsymbol{U}_1$, as described by the lemma. For $1 \leq i \leq 4k$, and $P_i'$ having ends $w, w' \in V_1''$, we prolong the path $P_i'$ to $P_i^+$ by adding the paths $Q_w^+$ and $Q_{w'}^+$. The new paths $P_1^+, \ldots, P_{4k}^+$ are clearly pairwise disjoint, and having both ends in $V(\boldsymbol{P})$. Suppose that $P_1'$ is the path closest in $\boldsymbol{U}_1$ to the root $r$. Then the cycle $C \subseteq P_1^+ \cup \boldsymbol{P}$ bounds an open region $R$ such that $T(P_i^+) \setminus T(\boldsymbol{P}) \subset R$ for $2 \leq i \leq 4k$, and that $R \subset X(\boldsymbol{M}_b)$ by the choice of $V_1''$. Since no edge incident with an internal vertex of $\boldsymbol{P}$ is in $I(\boldsymbol{M}_b)$, and since $\boldsymbol{P}$ and all $P_i^+$ are uncrossed, the sets $V(P_i^+)$, $1 \leq i \leq 4k$ are cuts in $\boldsymbol{G}$. It is now easy to verify that, indeed, $(P_1^+, P_2^+, \ldots, P_{4k}^+)$ is a cutting sequence in $\boldsymbol{G}$.

Consider the case when (N4)(a) or (b) happens. (Those two cases are essentially the same for the purpose of this proof.) We may apply the same construction as in the previous case, the only difference is that we consider paths ending in the vertex $v$ rather than on $\boldsymbol{P}$. So we obtain a sequence $P_1^+, \ldots, P_{4k}^+$ in $\boldsymbol{G}$ in the same way as above, but now each $P_i^+$ is a cycle in $\boldsymbol{G}$. All cycles $P_i^+$, $1 \leq i \leq 4k$ are sharing the vertex $v \in V(\boldsymbol{M}_b)$ defined by (N4)(a) or (b), but they are pairwise disjoint elsewhere. Moreover, all $P_i^+$ are contained in $X(\boldsymbol{M}_b) \cup \{v\}$.

Now, the assumptions of Lemma 6.2 are satisfied for $(\boldsymbol{M}_0)$ and the multicycle $\boldsymbol{N}_1^o$ formed by $P_1^+$ with $P_2^+$ in the exterior face. Hence there is a multicycle $\boldsymbol{N}_1 \preceq \boldsymbol{N}_1^o$ in $\boldsymbol{G}$ such that $(\boldsymbol{M}_0, \boldsymbol{N}_1)$ is a 2-nesting sequence. (We do not require the sequence to be strong.) Since the assumptions of the lemma are still satisfied for $(\boldsymbol{M}_0, \boldsymbol{N}_1)$ and a multicycle formed by $P_2^+$, and so on, we may repeat our argument for $P_2^+, \ldots, P_{3k-1}^+$. Finally, we get a $(3k-1)$-nesting sequence $(\boldsymbol{M}_0, \boldsymbol{N}_1, \ldots, \boldsymbol{N}_{3k-1})$ in $\boldsymbol{G}$. □

The whole proof is now finished. ∎

***Proof of Theorem 1.1.*** Let us suppose that there exists a 2-connected graph $\boldsymbol{G}$ contradicting the statement – i.e. $k$-crossing-critical, drawn in the plane with $k$ crossings, and containing a subdivision of a binary tree of height $f(k)$. Let $\boldsymbol{M}_0$ be the subgraph of $\boldsymbol{G}$ consisting of all crossed edges and their ends. Since there are $2^{f'(k)} > 4k \geq |V(\boldsymbol{M}_0)|$ disjoint trees in $\boldsymbol{G}$ that are subdivisions of a binary tree of height $f(k) - f'(k) = f_0(k)$, some of these trees, say $\boldsymbol{U}$, is disjoint from $\boldsymbol{M}_0$. By Lemma 6.1, $(\boldsymbol{M}_0)$ is a strong 0-nesting sequence. Then we repeatedly apply Lemma 8.1, until we get (after at most $3k-1$ steps) a contradiction to the existence of $\boldsymbol{G}$ by Lemma 4.2 or by Lemma 5.1.

So let us drop the connectivity assumption now, and suppose that $\boldsymbol{G}$ is an arbitrary $k$-crossing-critical graph that is drawn in the plane with $k$ crossings. The following is an easy observation:

*Claim 1.* Let $\boldsymbol{H}_1, \boldsymbol{H}_2$ be two graphs such that $|V(\boldsymbol{H}_1) \cap V(\boldsymbol{H}_2)| \leq 1$. Then $\mathrm{cr}(\boldsymbol{H}_1 \cup \boldsymbol{H}_2) = \mathrm{cr}(\boldsymbol{H}_1) + \mathrm{cr}(\boldsymbol{H}_2)$.

We decompose $\boldsymbol{G}$ into 2-connected components $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_n$. If some of the components is an isolated vertex, then it has no significance for our problem, so we discard it. Then, for $k_i = \mathrm{cr}(\boldsymbol{G}_i)$, $k_1 + \ldots + k_n = k$ holds by inductive application of Claim 1. Moreover, all graphs $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_n$ must be crossing-critical, and hence, in particular, $k_i > 0$. Thus the largest subdivision of a binary tree that any of $\boldsymbol{G}_i$, $i \in \{1, \ldots, n\}$ may contain is of height less than $f(k_i)$. Altogether, the largest subdivision of a binary tree in $\boldsymbol{G}$ is of height less than $f(k_1) + \ldots + f(k_n) + \lceil \log_2 n \rceil < f(k_1) + 1 + \ldots + f(k_n) + 1 < f(k)$. ∎

## 9 Final Remarks

A natural question arising in connection with Theorem 1.1 is whether the bound $f(k)$ must depend on $k$ at all. A simple answer is given by the complete graph $\boldsymbol{K}_n$ – it is crossing-critical for $n \geq 5$ from edge-transitivity, $\mathrm{cr}(\boldsymbol{K}_n)$ obviously tends to infinity with $n$, and $\boldsymbol{K}_n$ contains arbitrarily large binary tree for big $n$. In fact, we can get a much better lower bound on $f(k)$, as proved in [10].

**Theorem 9.1.** *Let $f$ be the function from Theorem 1.1, and let $k \geq 3$. Then $f(k) \geq k+3$, or $f(k) \geq k$ if we consider only simple 3-connected graphs.*

We are not going to give any conjecture about the behavior of $f(k)$ (other than what was proved here), but we think that the right order of magnitude is closer to the linear lower bound than to the upper bound $O(k^3 \log k)$.

Another question reader may ask is whether the proof can be extended to other surfaces than the plane. That is not clear at this moment. It seems to be possible to extend the definition of the nesting sequence to other (orientable) surfaces, and to carry on the arguments from Lemma 8.1 using homotopy classes for the surface. However, a problem is that the proofs of Lemma 4.2 and Lemma 5.1 completely fail on other surfaces. So we leave this question open.

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