Structure and Generation of Crossing-critical Graphs

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joint work with

Zdeněk Dvořák and **Bojan Mohar**

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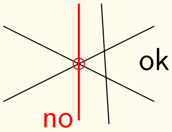

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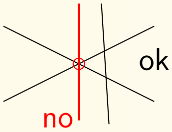

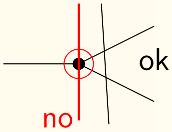

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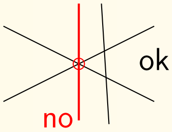

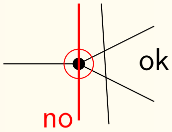

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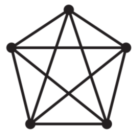
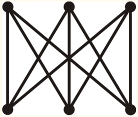
Definition. Graph H is *c-crossing-critical* if
$cr(H) \geq c$ and $cr(H - e) < c$ for all edges $e \in E(H)$.

We study crossing-critical graphs to understand what structural properties force the crossing number of a graph to be large.

Some starting examples

- Kuratowski (30): The only 1-crossing-critical graphs K_5 and $K_{3,3}$.





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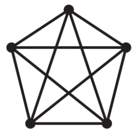

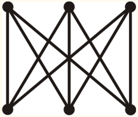

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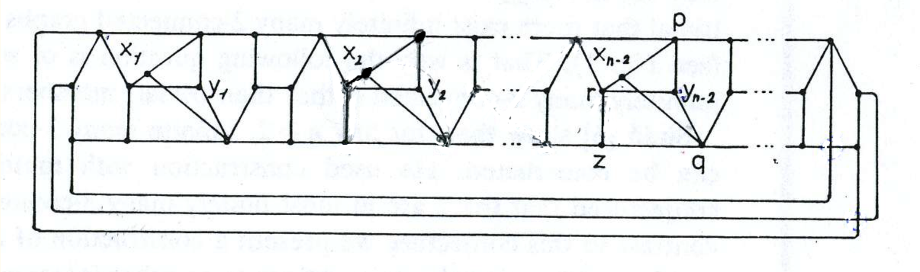
- Širáň (84), Kochol (87): Infinitely many c-crossing-critical graphs for every $c \geq 2$, even simple 3-connected.



And some more recent constructions

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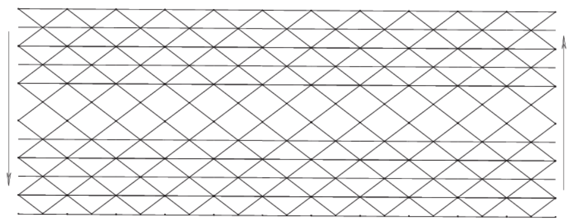

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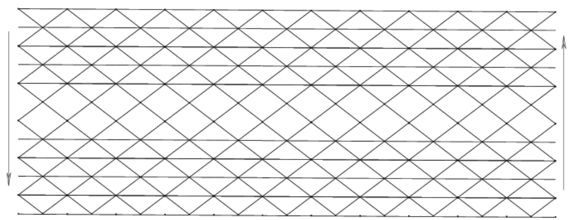
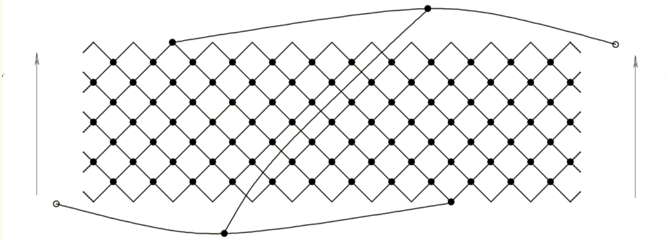
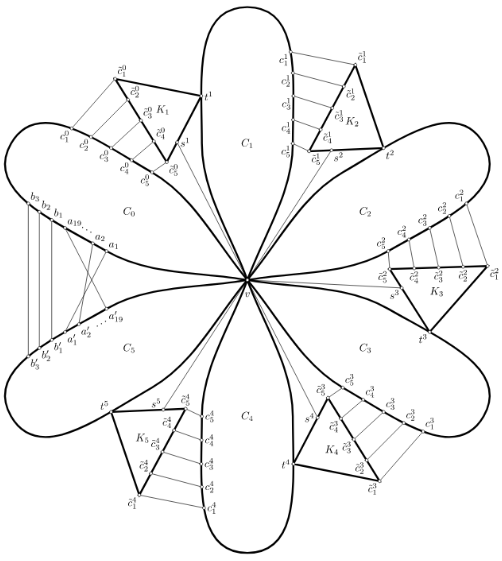


- Hliněný (02):



. . . and a bit of surprise

- Dvořák, Mohar (10): A c-crossing-crit. graph with unbounded degree, $c \geq 171$.



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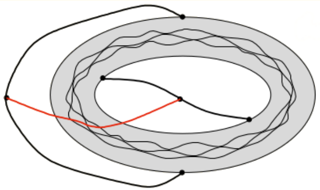
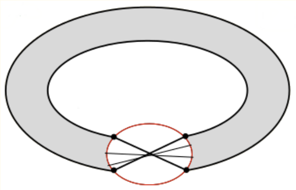
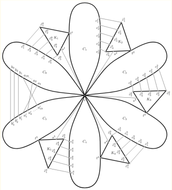
- Dvořák, Hliněný, Mohar, Postle (11, not published):
  A c-crossing-critical graph cannot contain a deep nest, and so it has bounded dual diameter.

What kinds of crossing-critical graphs do we have?

Informally, "thin and long" bands, joined together, and huge faces around...

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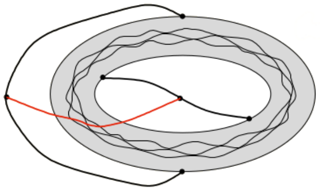

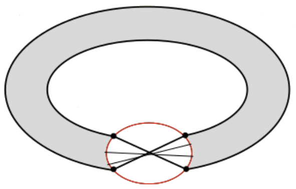

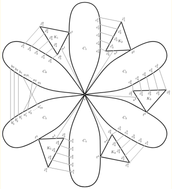

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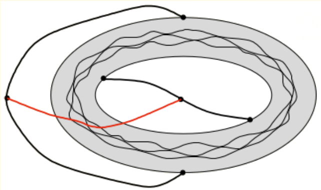

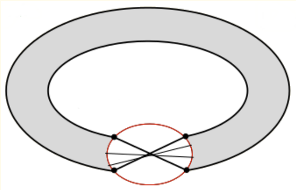

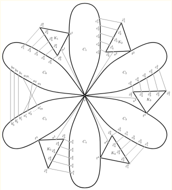

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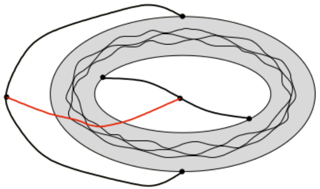 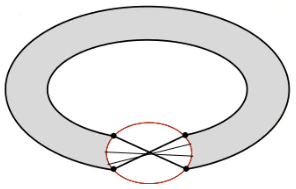 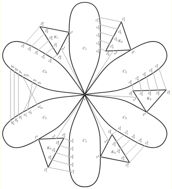

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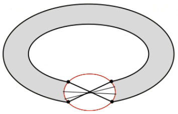

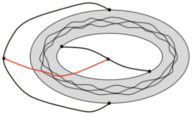

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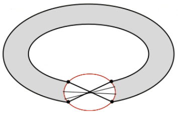

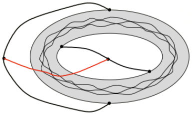

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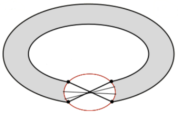

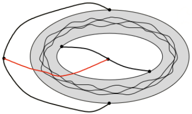

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Prove that such shortening preserves crossing-criticality.

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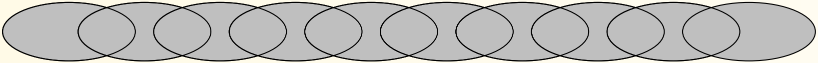

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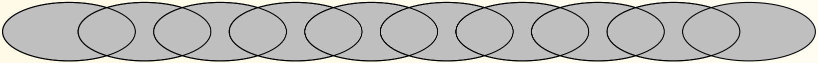

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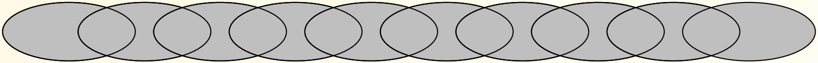

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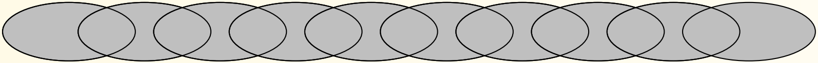

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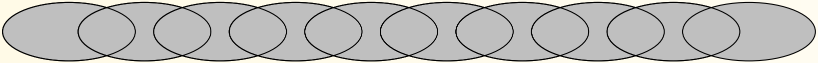

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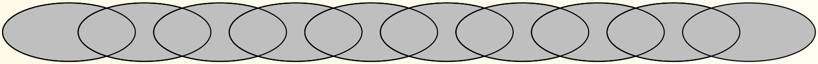

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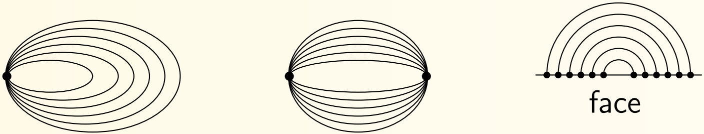

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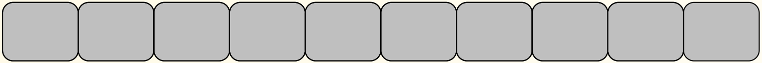
a) "Long band" → consider shelled bands, shelled fans, and necklaces.



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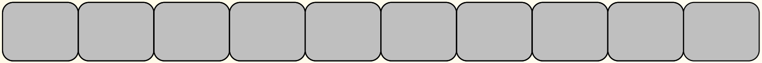

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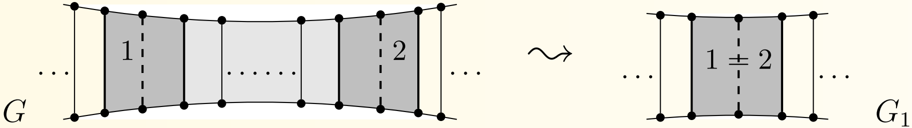

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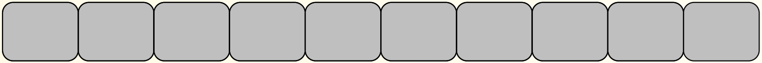

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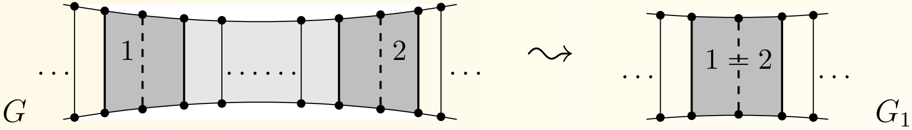
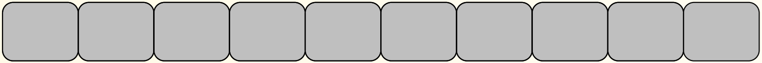


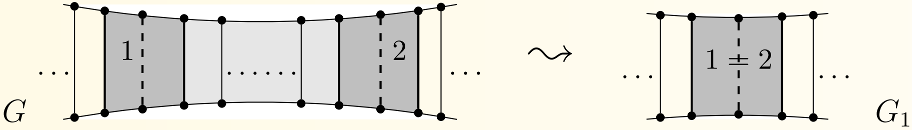
c) Use further repetitions of this local picture around to argue that c-crossing-criticality is preserved:

3.2 Removing and inserting tiles

a) "Long band" $\rightarrow$ consider shelled bands, shelled fans, and necklaces.



b) Repeated isomorphic sections $\rightarrow$ overlay, and forget the stretch betw.



c) Use further repetitions of this local picture around to argue that c-crossing-criticality is preserved:

– G_1 drawn with $< c$ crossings $\rightarrow$ can expand with no new crossing,

– (more difficult) $G - e$ drawn with $< c$ crossings $\rightarrow$ can modify and shrink to $G_1 - e$ with no new crossing.

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Thank you for your attention.