Structure and Generation of Crossing-critical Graphs, I.

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joint work with

Zdeněk Dvořák and **Bojan Mohar**

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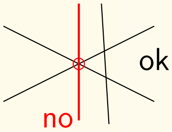

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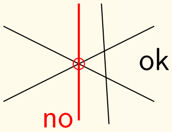

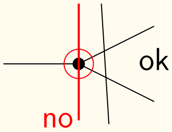

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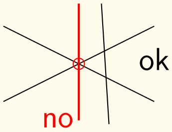

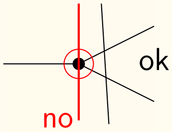

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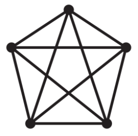
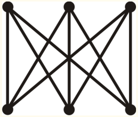
Definition. Graph H is *c-crossing-critical* if
$cr(H) \geq c$ and $cr(H - e) < c$ for all edges $e \in E(H)$.

We study crossing-critical graphs to understand what structural properties force the crossing number of a graph to be large.

Some starting examples

- Kuratowski (30): The only 1-crossing-critical graphs K_5 and $K_{3,3}$.





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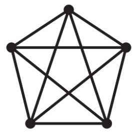
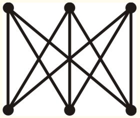

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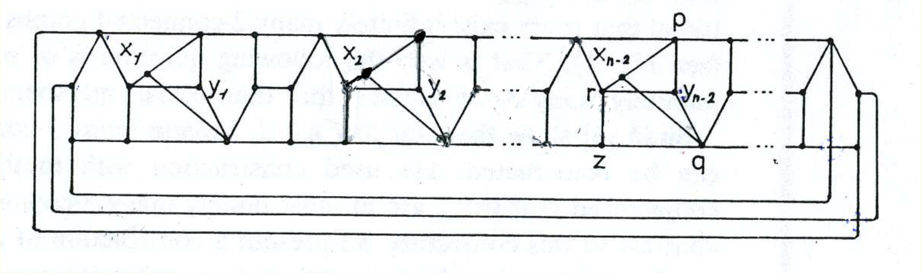
- Širáň (84), Kochol (87): Infinitely many c-crossing-critical graphs for every $c \geq 2$, even simple 3-connected.



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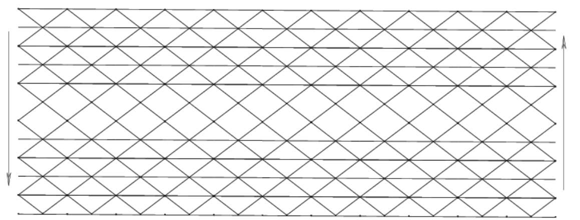

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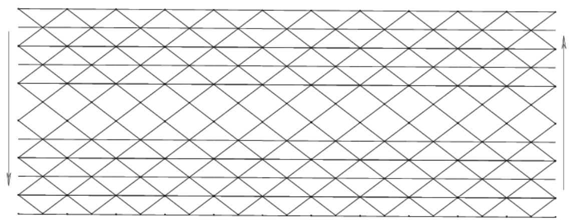
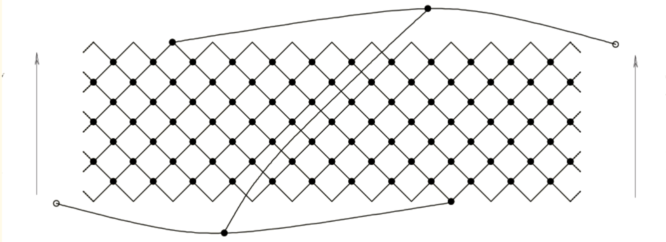


- Hliněný (02): "drop" by 1, but having *planarizing edge*.



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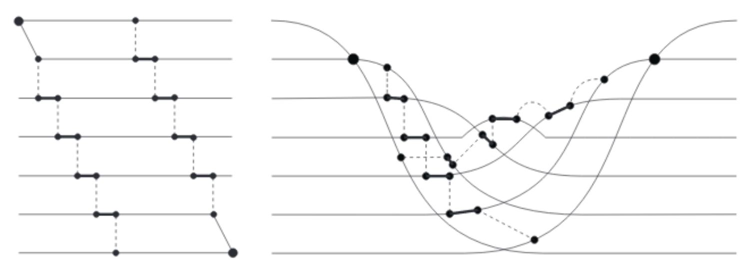
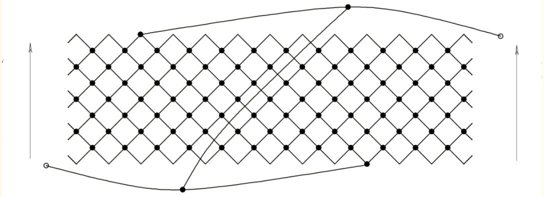
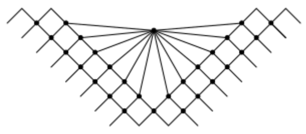
- Excluding average degree 3 via Graph minors...
- Excluding avg. deg. 6 – Hernández-Vélez, Salazar and Thomas (12).
- Getting average degree close to 3 – Bokal's (10) staircase strip.



- Construction with arbitrary even degrees – PH (11),





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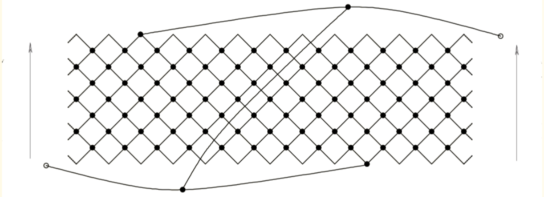
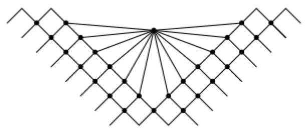
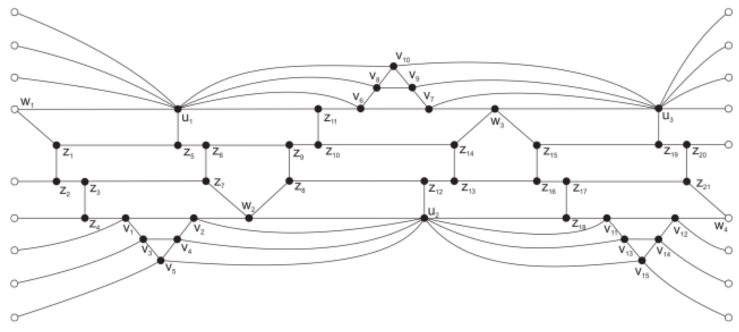


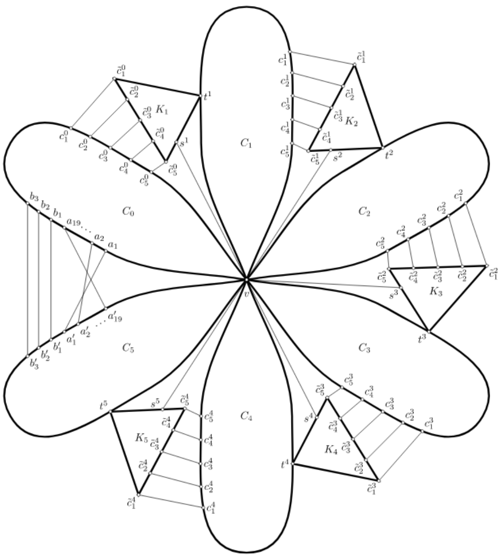


- and with arbitrary odd degrees – Bokal, Bračič, Derňár, PH (15).



. . . and a bit of surprise

- Dvořák, Mohar (10): A c-crossing-crit. graph with unbounded degree, $c \geq 171$.



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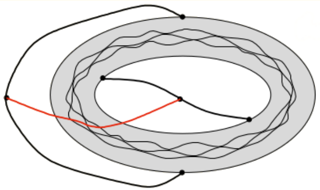
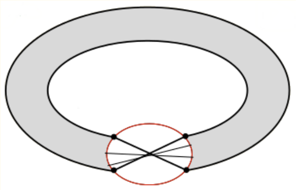
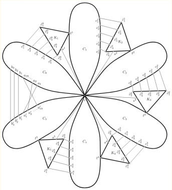
- Dvořák, Hliněný, Mohar, Postle (11, not published):
  A c-crossing-critical graph cannot contain a deep nest, and so it has bounded dual diameter.

What kinds of crossing-critical graphs do we have?

Informally, "thin and long" bands, joined together, and huge faces around. . .







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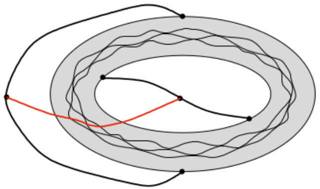
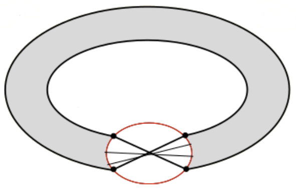
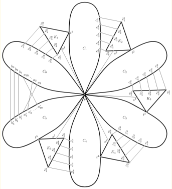

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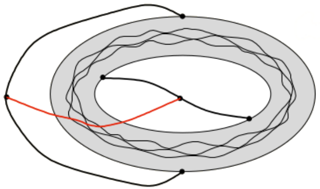

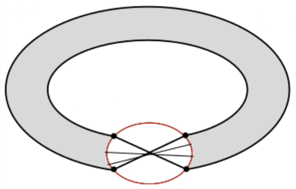

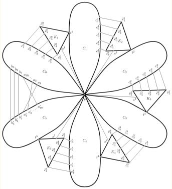

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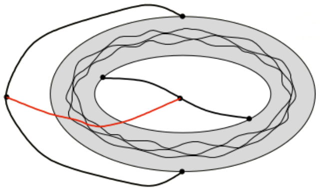
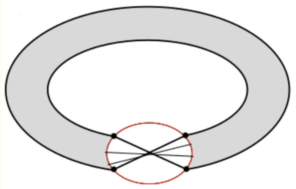
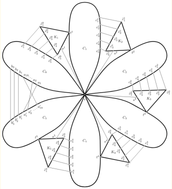

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III. There are finitely many well-defined building bricks that can produce all c-crossing-critical graphs from a finite set of base graphs.

Once again, with an informal explanation

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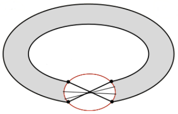

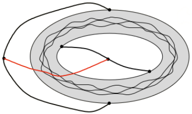

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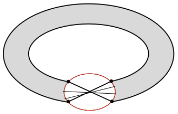

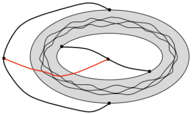

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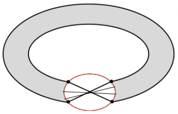
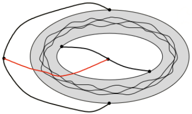


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4 To be continued. . .

by Zdeněk