On the Crossing Number of Almost Planar Graphs

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joint work with **Gelasio Salazar**

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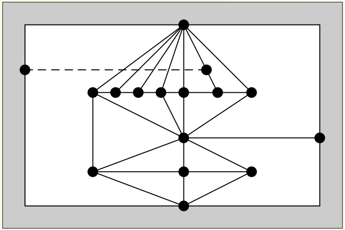
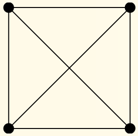


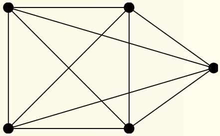
Overview

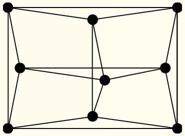
1 Drawings and the Crossing Number

Definition. *Drawing of a graph G*:

- The vertices of G are distinct points,
  and every edge $e = uv \in E(G)$ is a simple curve joining u to v.
- No edge passes through another vertex,
  and no three edges intersect in a common point.







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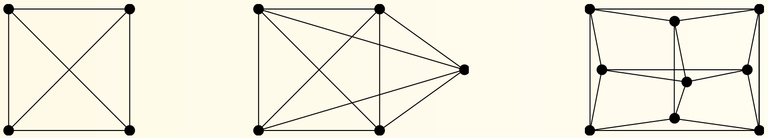

Definition. *Crossing number* $\mathrm{cr}(G)$
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Importance – in VLSI design [Leighton et al], graph visualization, etc.

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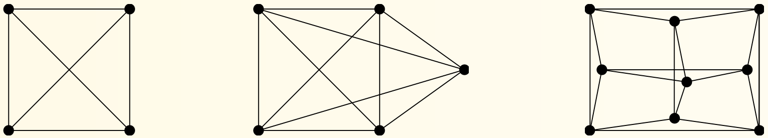

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Warning. There are slight variations of the definition of crossing number, some giving different numbers! (Like counting odd-crossing pairs of edges.)

Computational complexity

Remark. It is (practically) very hard to determine crossing number.

Observation. The problem $\textsc{CrossingNumber}(\leq k)$ is in NP:
Guess a suitable drawing of G, then replace crossings with new vertices, and test planarity.

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Theorem 1. [**Garey and Johnson**, 1983] $\textsc{CrossingNumber}$ *is* NP*-hard.*

Theorem 2. [**Grohe**, 2001] $\textsc{CrossingNumber}(\leq k)$ *is in* FPT *with parameter* k*, i.e. solvable in time* $O(f(k) \cdot n^2)$*.*

Theorem 3. [**PH**, 2004] $\textsc{CrossingNumber}$ *is* NP*-hard even on simple* 3*-connected cubic graphs.*

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Theorem 3. [**PH**, 2004] *$\textsc{CrossingNumber}$ is NP-hard even on simple 3-connected cubic graphs.*

Question 4. [PH, GS / Mohar, 2006] Is it an NP–hard problem to compute the crossing number of an apex graph?

2 Edge-insertion Heuristic

(Seemingly) best general practical heuristic approach to crossing minimization:

- Delete from G some (small set of) edges F, so that $G' = G - F$ is planar.
- Take an edge $f \in F$ and a suitable planar embedding of G', and insert f back to G' with the smallest number of crossings.
- Make $G' + f$ planar G'' by replacing the crossings with new vertices, and iterate the process with G'' and $F \setminus \{f\}$...

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This heuristic, in turn, outlines the following problem:

Definition. The problem of (one-edge) BRIDGINGMINIMIZATION has

Input: a planar graph G and two nonadjacent vertices u, v of G,

Problem: find a planar drawing of G such that the (new) edge uv can be inserted to G with the minimum number of crossings.

That problem has got a really nice solution!

Theorem 5. [**Gutwenger, Mutzel, Weiskircher**, 2001]
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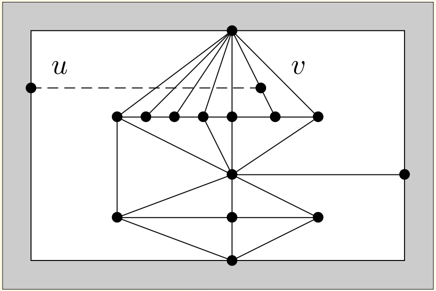
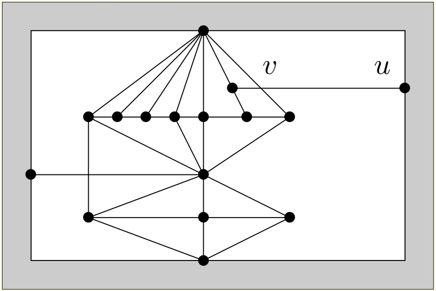
However, the answer is not so useful for the original problem. . .

Fact. [**Farr**, 2005] A solution to one-edge bridging minimization (left) can be arbitrarily far from the crossing number (right).





3 Crossing on Almost-planar Graphs

Our main **new contribution** is the following result:

Theorem 6. *Let G be a planar graph and u, v nonadjacent vertices of G. Then the bridging minimization problem on G and uv has a solution with*

$$\mathbf{br}(G, uv) \leq \Delta(G) \cdot \mathbf{cr}(G + uv)\,.$$

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Almost-planar – removing one edge leaves a planar graph.

Hence, for almost planar graphs of bounded degree, the algorithm of Gutwenger, Mutzel, and Weiskircher makes a

constant-factor approximation of the crossing number.

Some proof ideas

- **What is our situation?**

  Having a graph G with edge $e = uv$ such that $G - e$ is planar, and a crossing-optimal drawing G' of G in which $G' - e$ is not plane.

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- **What can we do now?**

  Delete (few, actually $\leq \mathrm{cr}(G')$) edges F to make $G' - F$ plane.
  Insert the edges of F back one-by-one, introducing $\leq \Delta$ new crossings on e for each one.

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- **Whitney flipping** – the tool to use:

  Flipping – on a 2-cut, re-embed one side with its mirror image.

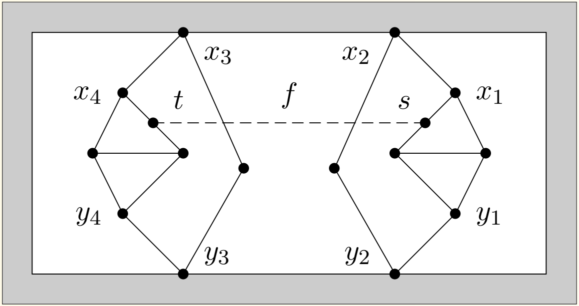
  Every two embeddings of the same (2-connected) planar graph can be transformed to each other via Whitney flippings.

  Hence we follow a sequence of flippings that transforms

  $$(G' - e - F) \text{ into } (G - e - F).$$

- **Whitney flippings** continued...

  However, many flippings might be needed to insert even one edge of F back, like the example with four flippings:



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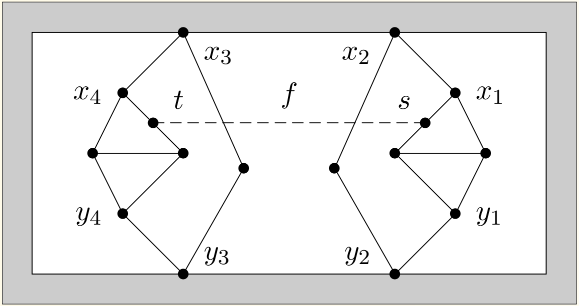

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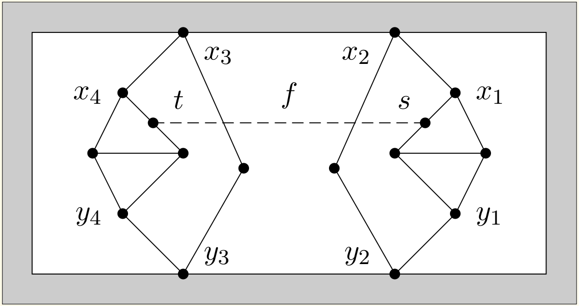
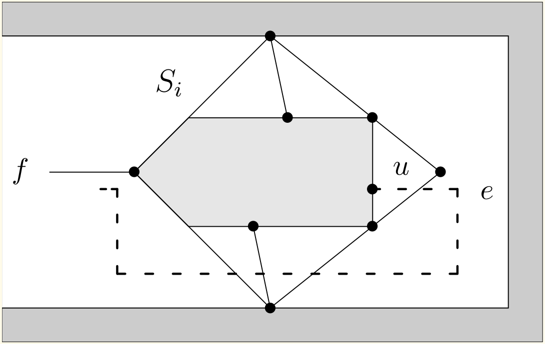


One flipping might introduce up to $\Delta(G)/2$ new crossings on e !?

Firstly, only those fippings that separate both ends of f, and both ends of e, from each other are relevant.

Secondly, only two of those flippings really contribute new crossings on e.

- **Whitney flippings** for third, an illustration:



Iterating this process with each edge of F, we get the bound

$$\underline{\mathrm{br}(G-e,e) \le \Delta(G)\cdot|F| \le \Delta(G)\cdot \mathrm{cr}(G)}\ .$$

■

4 Crossing-Critical Graphs

One more theoretical thought...

What forces high crossing number?

- Many edges – cf. Euler's formula, and some strong enhancements [Ajtai, Chvátal, Newborn, Szemeredi, 1982; Leighton].
- Structural properties (even with few edges) – but what exactly?

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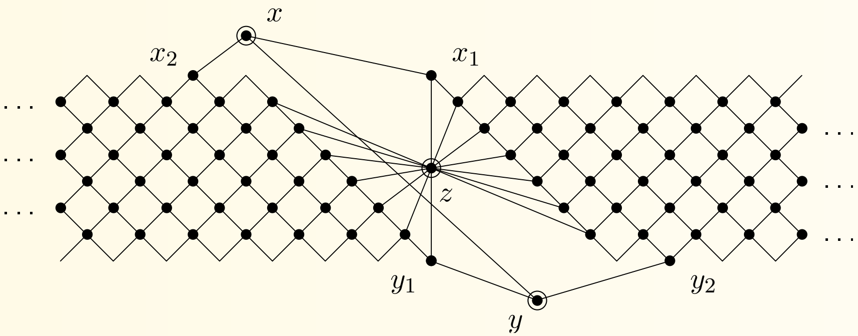
- Structural properties (even with few edges) – but what exactly?

Definition. Graph H is *k-crossing-critical*
– $\mathrm{cr}(H) \geq k$ and $\mathrm{cr}(H-e) < k$ for all edges $e \in E(H)$.

We study crossing-critical graphs to understand what structural properties force the crossing number of a graph to be large.

The exact crossing number problem seems to be nontrivial even on projective (and) almost-planar graphs!

Nontriviality is witnessed by a rich family of projective almost-planar k-crossing-critical graphs here...



Conclusions

- We have proved that, for almost planar graphs of bounded degree, the algorithm of Gutwenger, Mutzel, and Weiskircher gives an efficient constant-factor approximation of the crossing number.
- We have demonstrated nontriviality of the crossing number problem on almost-planar graphs.

Conclusions

- We have proved that, for almost planar graphs of bounded degree, the algorithm of Gutwenger, Mutzel, and Weiskircher gives an efficient constant-factor approximation of the crossing number.
- We have demonstrated nontriviality of the crossing number problem on almost-planar graphs.
- **The message**:

  We understand really little about the crossing number problem if we cannot solve it exactly even on almost-planar graphs!

  Can we get any reasonable FPT algorithm for crossing number based on "how far" the graph is from planarity?