On Colourability of Polygon Visibility Graphs

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joint work with

Onur Çağırıcı and Bodhayan Roy

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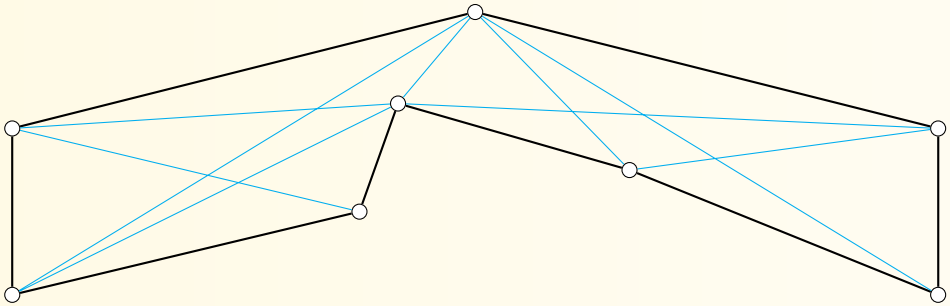

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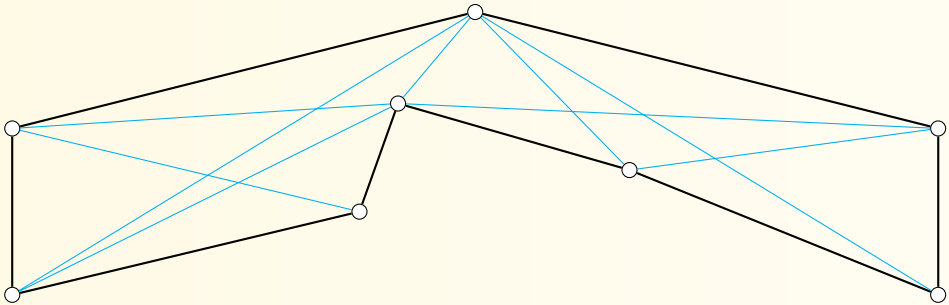

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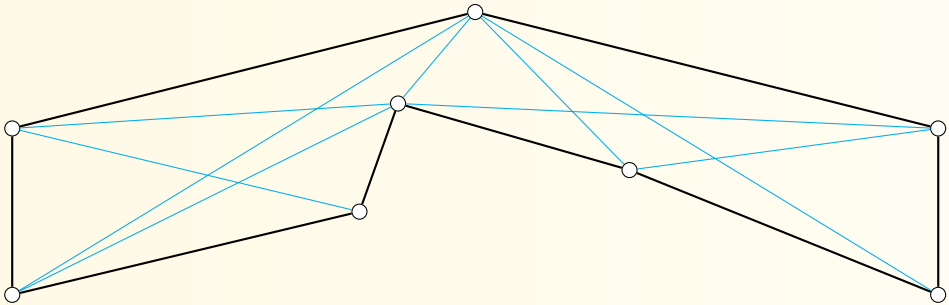

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- and more recent *conflict-free chromatic guarding* problems...

But,

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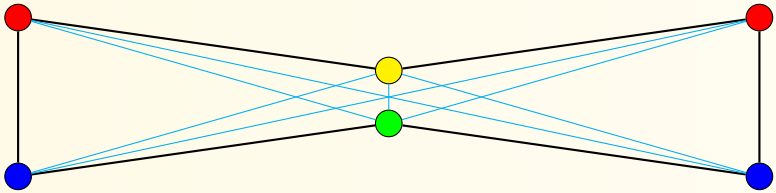
of the visibility graph of the given polygon:



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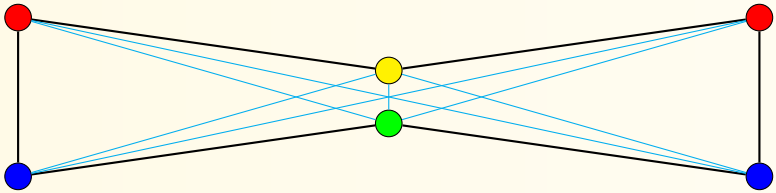

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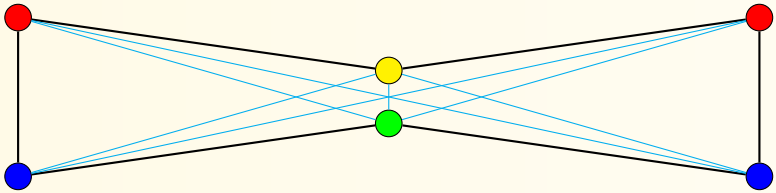

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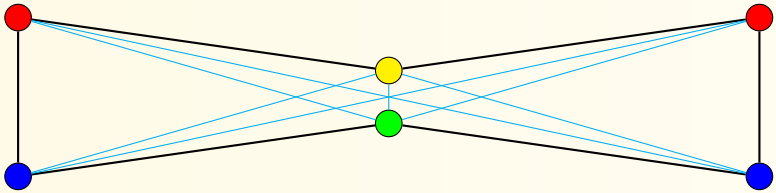

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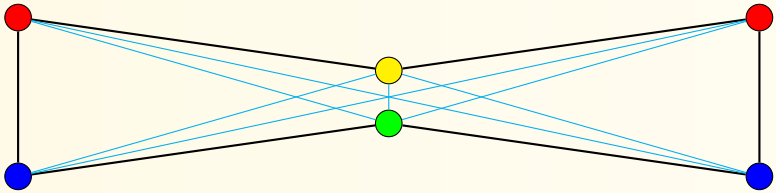

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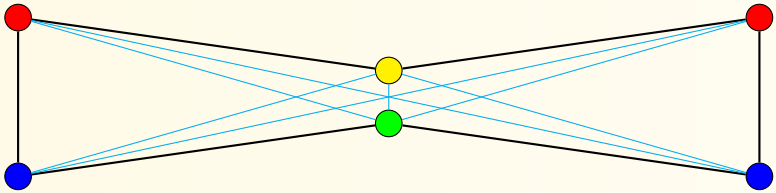

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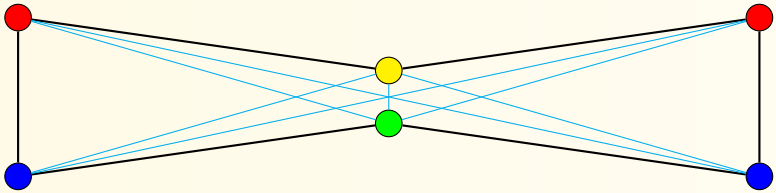

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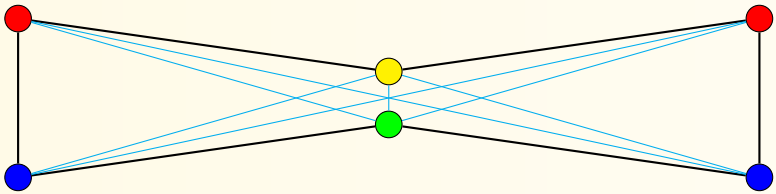

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- Specifically for our *polygon visibility* graphs;
  - 1995 [Lin and Skiena] complexity of k-colouring as open problem.

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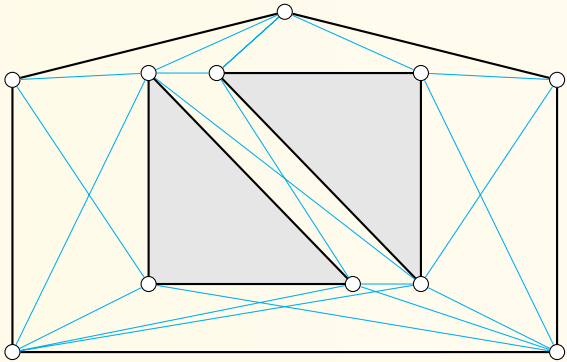
- 2019 even 4-colourability NP-complete for *polygons with holes*.



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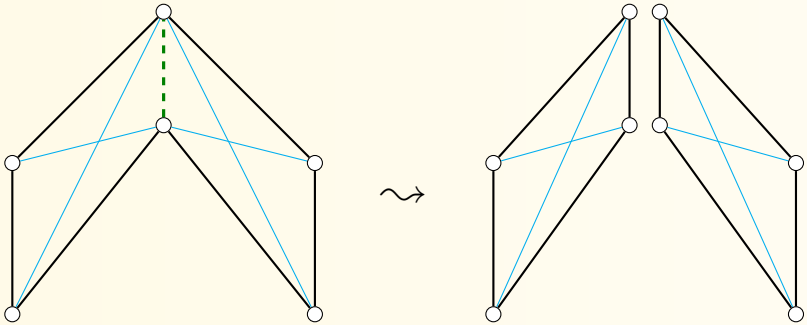
- *Reduced polygon*: cutting along uncrossed chords...



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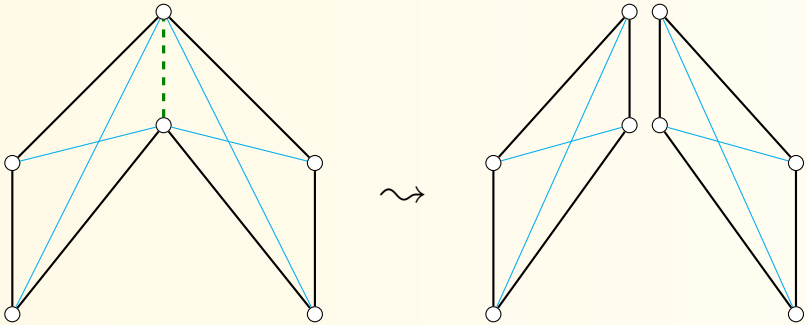

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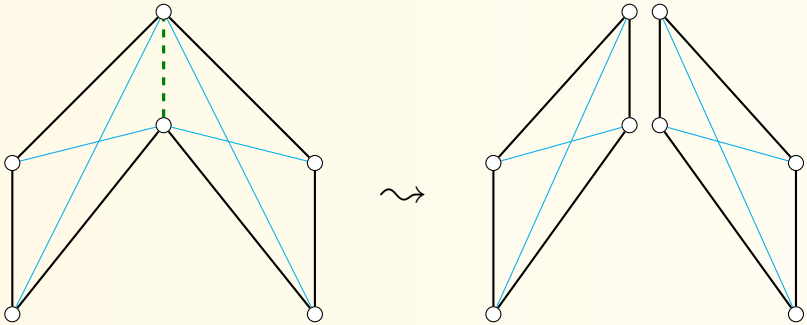

- **Theorem.** There is a unique 4-colouring (if any) of the visibility graph of a simple reduced polygon.

- **Corollary.** One can decide in polynomial time whether the given visibility graph of a simple polygon is 4-colourable (the polygon is not needed).

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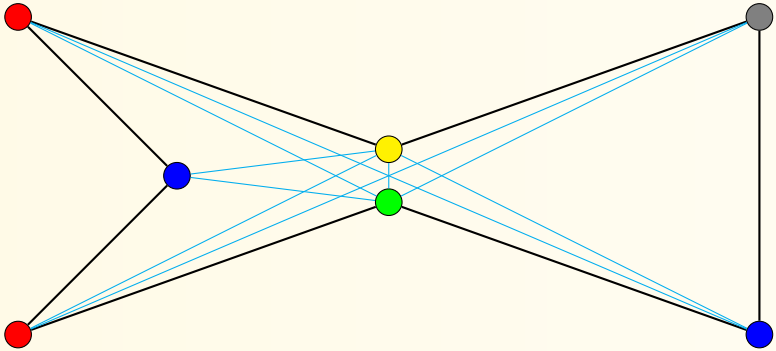
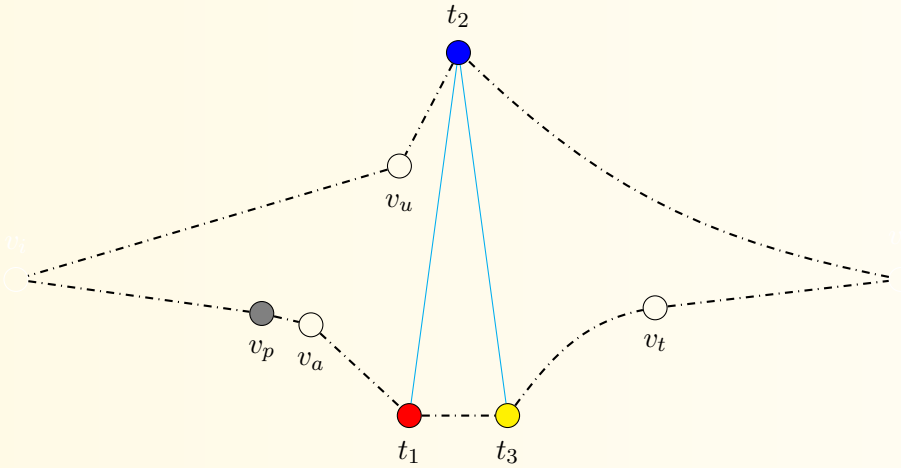


No...

Proof of unique 4-colouring

Only a brief picture sketch...



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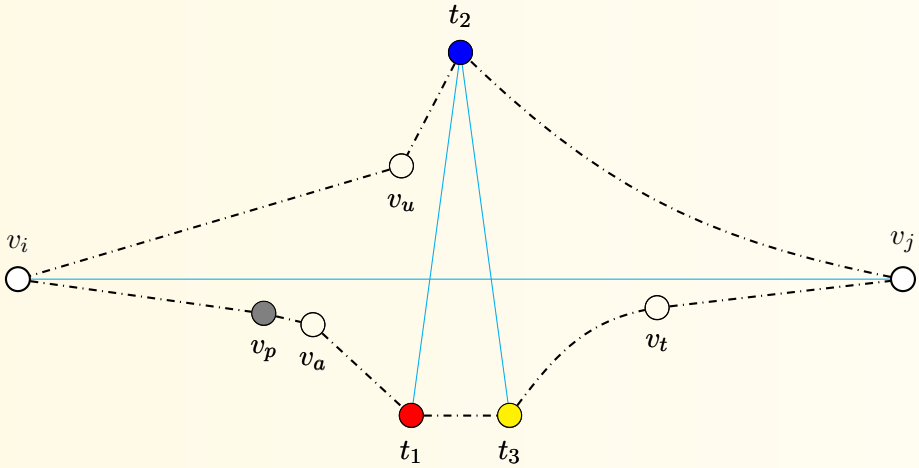

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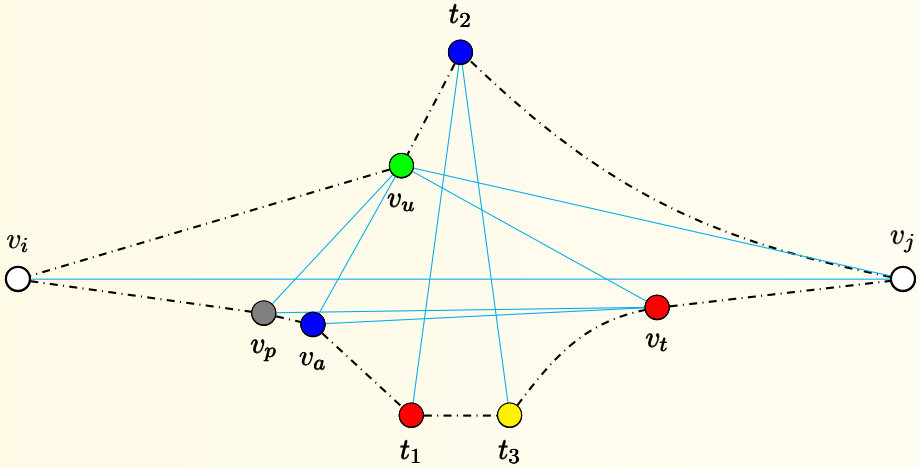

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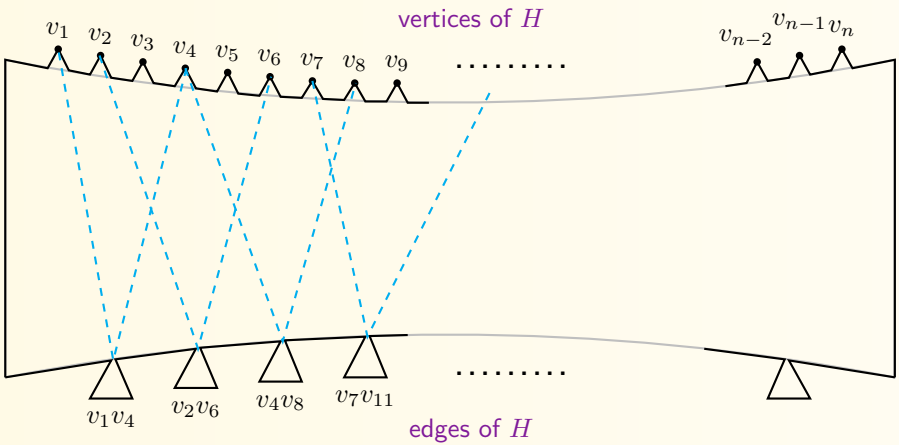

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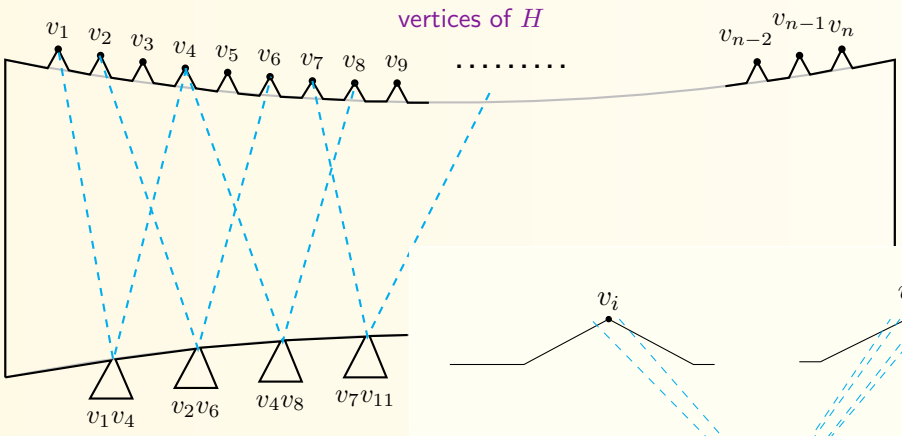

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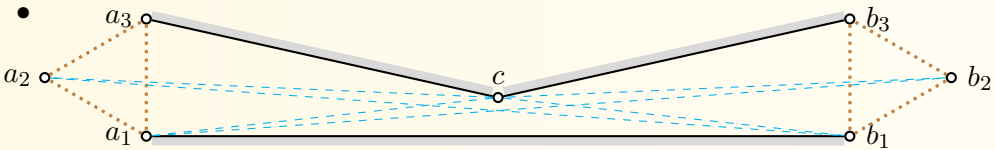

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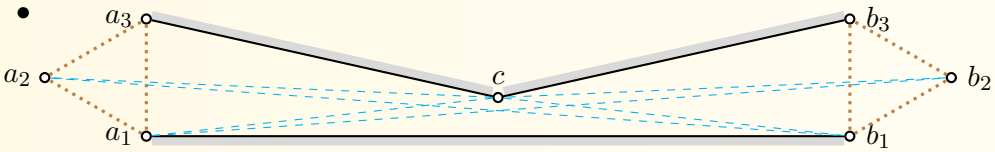
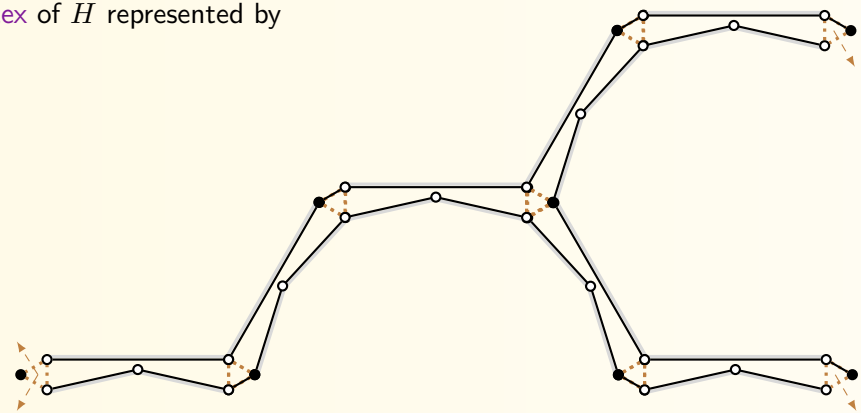
- 

- A vertex of H represented by



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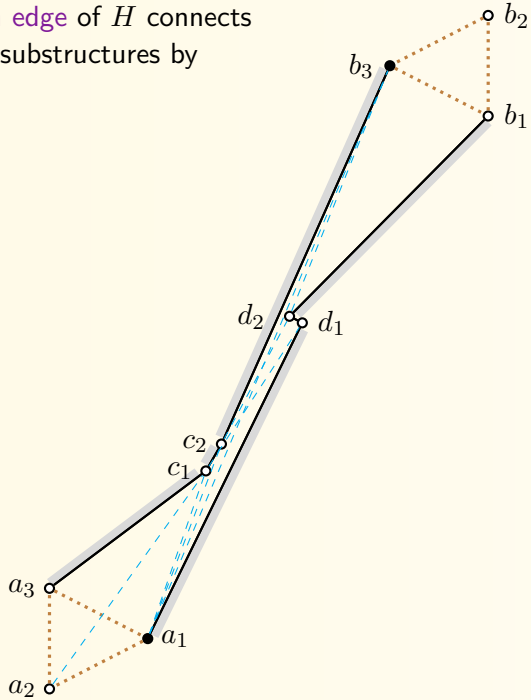

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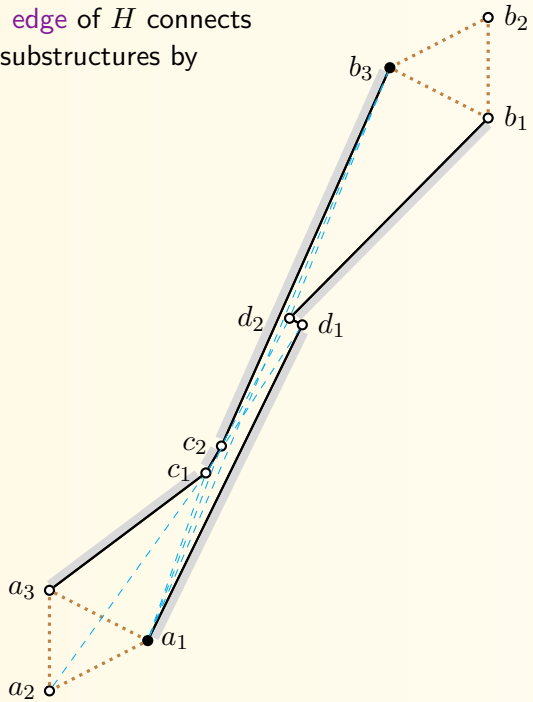

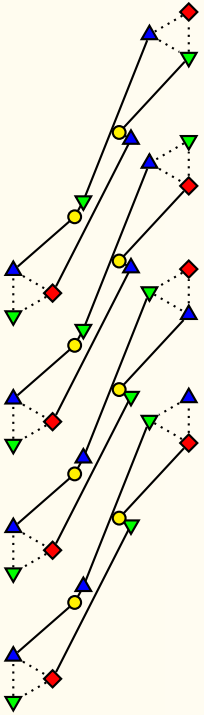

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- Full details arXiv:1904.08624.

Thank you for attention.