Unified Approach to Polynomial Algorithms on Graphs of Bounded (bi-)Rank-width

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Talk Outline

1 Measuring Graph "Width"

Motivation: Trees are easy to understand and to handle, so how "tree-like" our graphs are . . . , in some well-defined sense?

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- Many definitions have been studied so far,
  e.g. *tree-width*, *path-width*, *branch-width*, *DAG-width* . . .

- **Clique-width** – another graph complexity measure [Courcelle and Olariu], defined by operations on vertex–labeled graphs:
  - create a new vertex with label i,
  - take the disjoint union of two labeled graphs,
  - add all edges between vertices of label i and label j,
  - and relabel all vertices with label i to have label j.

Rank-Decompositions (a "better view" of clique-width)

- [Oum and Seymour, 03] Bringing the branch-decomposition approach to measure "complexity" of vertex subsets $X \subseteq V(G)$ via *cut-rank*:

$$\varrho_G(X) = \text{rank of } \begin{array}{cc} & V(G) - X \\ X & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \text{ modulo } 2$$

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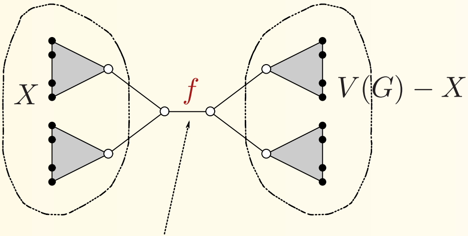

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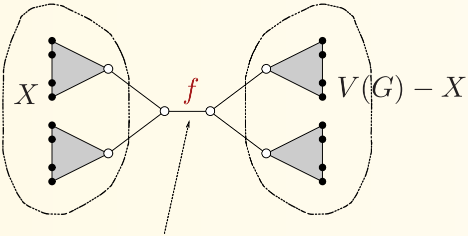

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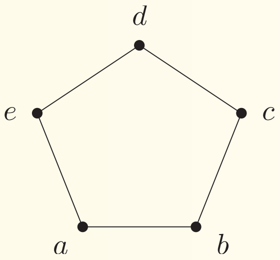
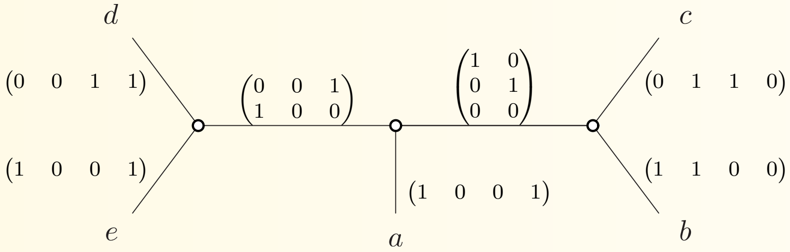
Rank-width $= \min_{\text{rank-decs. of } G} \max \{\text{width}(f) : f \text{ tree edge}\}$

An example. Cycle C_5 and its *rank-decomposition* of width 2:





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- [Oum and PH, 07] There is an *FPT algorithm* for computing an optimal rank-decomposition of a graph in time $O(f(t) \cdot n^3)$.
- And some new results suggest that algorithms designed on rank-decompositions run faster than those designed on clique-width expressions...

2 Dynamic Algorithms and Parse Trees

- A typical idea for a *dynamic algorithm* on a "tree-like" decomposition:
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  - Process this information bottom-up in the decomposition.
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- Combinatorial extensions of this concept appeared e.g. in the works [Abrahamson and Fellows, 93], [PH, 03], or [Ganian and PH, 08].

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- Consider the universe of graphs $\mathcal{U}_k$ implicitly associated with
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  - a *join operation* $G \oplus H$ acting on the boundaries of disjoint G, H.
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- Informally, the classes of $\approx_{\mathcal{P},k}$ capture all information about the property $\mathcal{P}$ that can "cross" our graph boundary of size k
(regardless of actual meaning of "boundary" and "join").

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- Considering a rooted ???-decomposition of a graph G, we build on the following correspondence:

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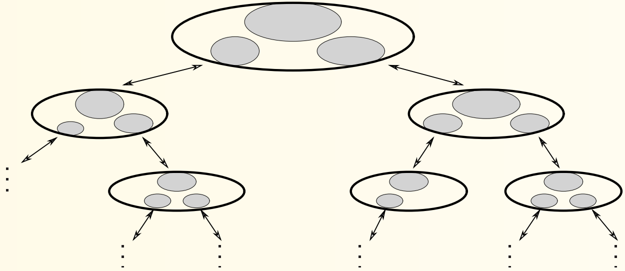
- This can be (visually) seen as...



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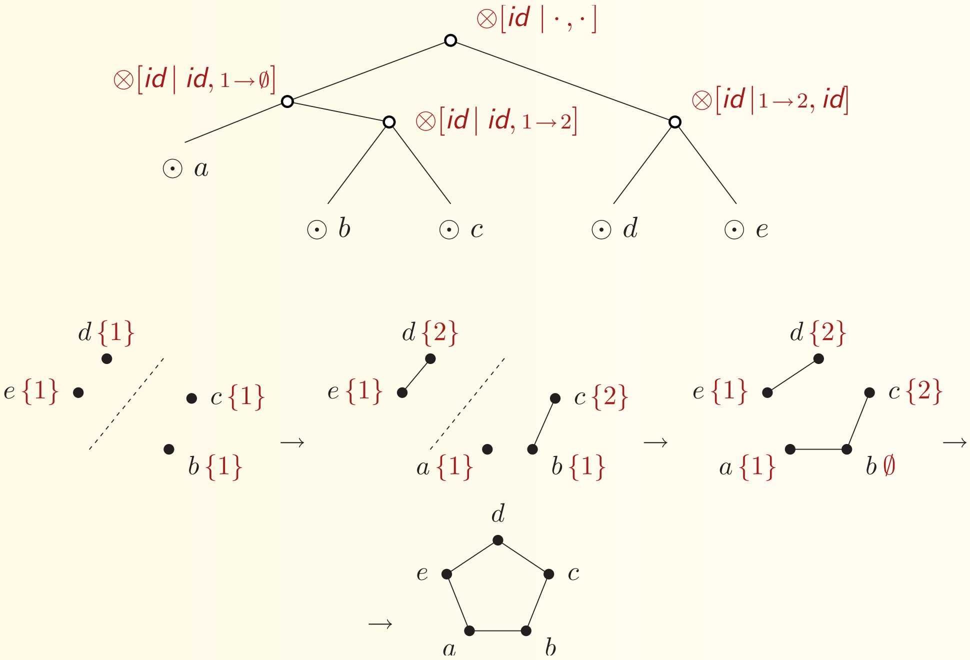
- Independently considered related notion of *R_t-join* decompositions by [Bui-Xuan, Telle, and Vatshelle, 08].

Parse tree. An example generating the cycle C_5 (of rank-width 2):



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- For parse trees, a straightforward generalization reads:

Theorem. (Analogy of [Myhill–Nerode])
$\mathcal{P}$ is accepted by a finite tree automaton on parse trees of boundary size $\leq k$
$\iff$ the *canonical equivalence* $\approx_{\mathcal{P},k}$ has finitely many classes on $\mathcal{U}_k$.

(Actually, this is a "metatheorem" which requires several more unspoken technical conditions on the parse trees to hold true...)

Extended canonical equivalence

$\boldsymbol{G_1 \approx_{\mathcal{P},k} G_2}$ for any $G_1, G_2 \in \mathcal{U}_k$ if and only if, for all $H \in \mathcal{U}_k$,

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- To apply this concept to predicates $\mathcal{P}(X_1, \dots)$ with free variables, we extend the universe $\mathcal{U}_k$ to *partially-equipped* graphs of boundary $\leq k$.

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- From that one easily concludes an older result:

Theorem. [Courcelle, Makowsky, and Rotics 00]

All *$LinEMSO$ graph optimization* problems (in MS_1 language – only vertices!) on the graphs of bounded rank-width t can be solved in FPT time $\boldsymbol{O(f(t)\cdot n)}$.

Core idea: In dynamic processing of the given parse tree, record optimal representatives of each class of the extended canonical equivalence $\approx_{\phi,t}$ ...

5 Unified Design Style of XP Algorithms

(XP: running in time $O(n^{f(k)})$, not FPT)

Starting point: For many problems $\mathcal{P}$, the number of classes of $\approx_{\mathcal{P},k}$ depends on the input size n ($\rightarrow$ likely no FPT algorithm exists).

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- and we can recombine the fragment enumerations efficiently.

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XP algorithm wrt. clique-width given by [Espelage, Gurski, and Wanke, 2001].

Theorem. Decide whether a graph G of rank-width t has a *Hamiltonian path* in time

$$O\left(|V(G)|^{\ell(t)}\right) \text{ where } \ell(t) = 4^{t+1} + O(1)$$

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  - and enumerated at every parse tree node as one multiset.
- Straightforward dynamic alg. processing then gives the result.

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Defective (ℓ, q)-colouring – partition the vertices into ℓ parts such that
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Considered recently by [Kolman, Lidický, and Sereni, 2009] wrt. tree-width.

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THANK YOU FOR YOUR ATTENTION