On Degree Properties of Crossing-critical Families of Graphs

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Abstract. Answering an open question from 2007, we construct infinite k-crossing-critical families of graphs which contain arbitrarily often vertices of any prescribed odd degree, for sufficiently large k. From this we derive that, for any set of integers D such that $\min(D) \geq 3$ and $3, 4 \in D$, and for all sufficiently large k there exists an infinite k-crossing-critical family such that the numbers in D are precisely the vertex degrees which occur arbitrarily often in this family. We also investigate what are the possible average degrees of such crossing-critical families.

Keywords: crossing number, tile drawing, degree-universality, average degree, crossing-critical graph.

1 Introduction

Reducing the number of crossings in a drawing of a graph is considered one of the most important drawing aesthetics. Consequently, great deal of research work has been invested into understanding of what forces the number of edge crossings in a drawing of the graph to be high. There exist strong quantitative lower bounds, such as the famous Crossing Lemma [1, 14]. However, the quantitative bounds typically show their strength only in dense graphs, while in the area of graph drawing we often deal with graphs having "small" number of edges.

The reasons for sparse graphs to have many crossings in any drawing are structural (there is a lot of "nonplanarity" in them). These reasons can be understood via so called *k-crossing-critical* graphs, which are the minimal graphs that require at least k edge crossings (the "minimal obstructions"). While there are only two 1-crossing-critical graphs, up to subdivisions—the Kuratowski graphs K_5 and $K_{3,3}$, it has been known already since Širáň's [19] and Kochol's [13]

⋆ This research was supported by the internationalisation of Slovene higher education within the framework of the Operational Programme for Human Resources Development 2007–2013 and by the Slovenian Research Agency project L7–5459.

⋆⋆ This research has been supported by the Czech Science Foundation under the project 14-03501S.

constructions that the structure of crossing-critical graphs is quite rich and non-trivial for any $k \geq 2$.

Although 2-crossing-critical graphs can be reasonably (although not easily) described [5], a full description for any $k \geq 3$ is clearly out of our current reach. Consequently, research has focused on interesting properties shared by all k-crossing-critical graphs (for certain k). Successfull attempts include, e.g., [7, 8, 10, 12, 17]. While we would like to establish as much specific properties of crossing-critical graphs as possible, the reality unfortunately seems to be against it. Many desired and conjectured properties of crossing-critical graphs have already been disproved by often complex and sophisticated constructions showing the odd behaviour of crossing-critical graphs on global scale, e.g. [6, 11, 9, 18].

We study properties of *infinite families* of k-crossing-critical graphs, for fixed values of k, since sporadic "small" examples of critical graphs tend to behave very wildly for every $k > 1$. Among the most studied such properties are those related to vertex degrees in the critical families, see [3, 6, 8, 11, 18]. Often the research focused on the average degree a k-crossing-critical family may have—this rational number clearly falls into the interval $[3, 6]$ if we forbid degree-2 vertices. It is nowadays known [8] that the true values fall into the open interval $(3, 6)$, and all the rational values there can be achieved [3].

In connection with the proof of bounded pathwidth for k-crossing-critical families [9, 10], it turned out to be a fundamental question whether k-crossing-critical graphs have maximum degree bounded in k. The somehow unexpected negative answer was then given by Dvořák and Mohar [6]. In 2007, Bokal noted that all the known (by that time) constructions of infinite k-crossing-critical families seem to use only vertices of degrees $3, 4, 6$, and he asked what other degrees can occur frequently often in k-crossing-critical families. Shortly after that Hliněný extended his previous construction [9] to include an arbitrary combination of any even degrees [11], for sufficiently large k.

Though, [11] answered only an (easier) half of Bokal's question, and it remained a wide open problem of whether there exist infinite k-crossing-critical families whose members contain many vertices of odd degrees greater than 5. Our joint investigation has recently led to an ultimate positive answer.

The contribution and new results of our paper can be summarized as follows:

- In Section 2, we review the tools which are commonly used in constructions of crossing-critical families.
- Section 3 presents the key new contribution—a construction of crossing-critical graphs with repeated occurrence of any prescribed odd vertex degree (Proposition 3.1 and Theorem 3.2).
- In Section 4, we combine the new construction of Section 3 with previously known constructions to prove the following: for any set of integers D such that $\min(D) \geq 3$ and $3, 4 \in D$, and for all sufficiently large k there exists an infinite k-crossing-critical family such that the numbers in D are precisely the vertex degrees which occur arbitrarily often in this family (Theorem 4.2).

– We then extend the previous results in Section 5 to include also an exhaustive discussion of possible average vertex degrees attained by our degree-restricted crossing-critical families (Theorem 5.1).
– Finally, in concluding Section 6 we pay special attention to 2-crossing-critical graphs, and list some remaining open questions.

2 Preliminaries

We consider *finite multigraphs* without loops by default (i.e., we allow multiple edges unless we explicitly call a graph *simple*), and use the standard graph terminology otherwise. The *degree* of a vertex v in a graph G is the number of edges of G incident to v (cf. multigraphs), and the *average degree* of G is the average of all the vertex degrees of G.

Crossing number. In a *drawing* of a graph G, the vertices of G are points and the edges are simple curves joining their endvertices. It is required that no edge passes through a vertex, and no three edges cross in a common point. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossing points of edges in a drawing of G in the plane. For $k \in \mathbb{N}$, we say that a graph G is *k-crossing-critical*, if $\mathrm{cr}(G) \geq k$ but $\mathrm{cr}(G - e) < k$ for each edge $e \in E(G)$.

Notice that a vertex of degree 2 in G is not relevant for a drawing of G and for the crossing number, and we will often replace such vertices by edges between their two neighbours. Since also vertices of degree 1 are irrelevant for the crossing number, it is quite common to assume minimum degree 3.

Degree-universality. The following terms formalize a vague notion that a certain vertex degree occurs frequently or arbitrarily often in an infinite family. For a set $D \subseteq \mathbb{N}$, we say that a family of graphs $\mathcal{F}$ is *D-universal*, if and only if, for every assignment of integers $\{m_d \mid d \in D\}$, there exists a graph $G \in \mathcal{F}$, such that G has at least m_d vertices of degree d for each $d \in D$. It follows easily that $\mathcal{F}$ has infinitely many such graphs.

Clearly, if $\mathcal{F}$ is D universal and $D' \subseteq D$, then $\mathcal{F}$ is also D'-universal. Thus for a particular family $\mathcal{F}$, we may focus our interest on the (unique) maximal set D for which $\mathcal{F}$ has this property—we then say that $\mathcal{F}$ is *D-max-universal*.

Tools for crossing-critical graphs. A principal tool used in many constructions of crossing-critical graphs are tiles. They were implicitly used already in the early papers by Kochol [13] and Richter–Thomassen [17], although they were formalized only later in the work of Pinnontoan and Richter [15, 16]. In our contribution, we use extension of their formalization from [3]. We refer also to the Appendix for a more detailed treatment of this useful toolbox.

A *tile* is a triple $T = (G, \lambda, \rho)$ where $\lambda, \rho \subseteq V(G)$ are two disjoint sequences of distinct vertices of G, called the *left and right wall* of T, respectively. A *tile drawing* of T is a drawing of the underlying graph G in the unit square such that the vertices of λ occur in this order on the left side of the square and those of ρ

in this order on the right side of it. The *tile crossing number* $\mathrm{tcr}(T)$ of a tile T is the smallest crossing number over all tile drawings of T.

For simplicity, in this brief exposition, we shall assume that all considered tiles satisfy $|\lambda| = |\rho| = w$ for a suitable universal constant $w \geq 2$ (though, a more general treatment is obviously possible). The *join of two tiles* $T = (G, \lambda, \rho)$ and $T' = (G', \lambda', \rho')$ is defined as the tile $T \otimes T' := (G'', \lambda, \rho')$, where G'' is the graph obtained from the disjoint union of G and G', by identifying ρ_i with λ'_i for $i = 1, \ldots, w$. Specially, if $\rho_i = \lambda'_i$ is a vertex of degree 2 (after the identification), we replace it with a single edge in G''. Since the operation $\otimes$ is associative, we can safely define the *join of a sequence of tiles* $\mathcal{T} = (T_0, T_1, \ldots, T_m)$ as $\otimes\mathcal{T} = T_0 \otimes T_1 \otimes \ldots \otimes T_m$. The *cyclization* of a tile $T = (G, \lambda, \rho)$, denoted by $\circ T$, is the ordinary graph obtained from G by identifying λ_i with ρ_i for $i = 1, \ldots, w$. The *cyclization of a sequence of tiles* $\mathcal{T} = (T_0, T_1, \ldots, T_m)$ is $\circ\mathcal{T} := \circ(\otimes\mathcal{T})$. Again, possible degree-2 vertices are replaced with single edges.

Let $T = (G, \lambda, \rho)$ be a tile. The *right-inverted* tile $T^{\updownarrow}$ is the tile $(G, \lambda, \bar{\rho})$ and the *left-inverted* tile ${}^{\updownarrow}T$ is the tile $(G, \bar{\lambda}, \rho)$, where $\bar{\lambda}$ and $\bar{\rho}$ denote the inverted sequences of λ, ρ. For a sequence of tiles $\mathcal{T} = (T_0, \ldots, T_m)$, let $\mathcal{T}^{\updownarrow} := (T_0, \ldots, T_m^{\updownarrow})$.

One can easily get [15]; for any tile T, $\mathrm{cr}(\circ T) \leq \mathrm{tcr}(T)$, and for every sequence of tiles $\mathcal{T} = (T_0, T_1, \ldots, T_m)$, $\mathrm{tcr}(\otimes\mathcal{T}) \leq \sum_{i=0}^{m} \mathrm{tcr}(T_i)$. On the other hand, corresponding lower bounds on the crossing number of cyclizations of tile sequences are also possible [3], under additional technical assumptions. A tile $T = (G, \lambda, \rho)$ is *planar* if $\mathrm{tcr}(T) = 0$. T is *perfect* if the following hold:

- $G - \lambda$ and $G - \rho$ are connected;
- for every $v \in \lambda$ there is a path from v to the right wall ρ in G internally disjoint from λ, and for every $u \in \rho$ there is a path from u to the left wall λ in G internally disjoint from ρ;
- for every $0 \leq i < j \leq w$, there is a pair of disjoint paths, one joining λ_i and ρ_i, and the other joining λ_j and ρ_j.

We are in particular interested in the following specialized result:

Theorem 2.1 ([3]). *Let $T_0, \ldots, T_m$ be copies of a perfect planar tile T, and $\mathcal{T} = (T_0, \ldots, T_m)$. Assume that, for some integer $k \geq 1$, we have $m \geq 4k - 2$ and $\mathrm{tcr}(\otimes(\mathcal{T}^{\updownarrow})) \geq k$. Then, $\mathrm{cr}(\circ(\mathcal{T}^{\updownarrow})) \geq k$.*

To lower-bound the tile crossing number (e.g., for use in Theorem 2.1), we use the following simple tool. A *traversing path* in a tile $T = (G, \lambda, \rho)$ is a path $P \subseteq G$ such that one end of P is in λ and the other in ρ, and P is internally disjoint from $\lambda \cup \rho$. A pair of traversing paths $\{P, Q\}$ is *twisted* if P, Q are disjoint and the mutual order of their ends in λ is the opposite of their order in ρ. Obviously, a twisted pair must induce a crossing in any tile drawing of T. A family of twisted pairs of traversing paths is called a *twisted family*.

Lemma 2.2 ([3]). *Let $\mathcal{F}$ be a twisted family in a tile T, such that no edge occurs in two distinct paths of $\cup\mathcal{F}$. Then, $\mathrm{tcr}(T) \geq |\mathcal{F}|$.*

The second tool for constructing crossing-critical families is the so called *zip product* [2, 3], which we introduce in a simplified setting [11]. For $i \in \{1, 2\}$, let G_i be a simple graph and let $v_i \in V(G_i)$ be its vertex of degree 3, such that $G_i - v_i$ is connected. We denote the neighbours of v_i by u^i_j for $j \in \{1, 2, 3\}$. The *zip product of* G_1 *and* G_2 *according to* v_1, v_2 and their neighbours, is obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding the three edges $u^1_1u^2_1$, $u^1_2u^2_2$, $u^1_3u^2_3$. The following is true in this special case:

Theorem 2.3 ([4]). *Let G be a zip product of G_1 and G_2 according to degree-3 vertices. Then,* $\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$. *Consequently, if G_i is k_i-crossing-critical for $i = 1, 2$, then G is $(k_1 + k_2)$-crossing-critical.*

3 Crossing-Critical Families with High Odd Degrees

We first present a new construction of a crossing-critical family containing many vertices of an arbitrarily prescribed odd degree (recall that the question of an existence of such families has been the main motivation for this research).

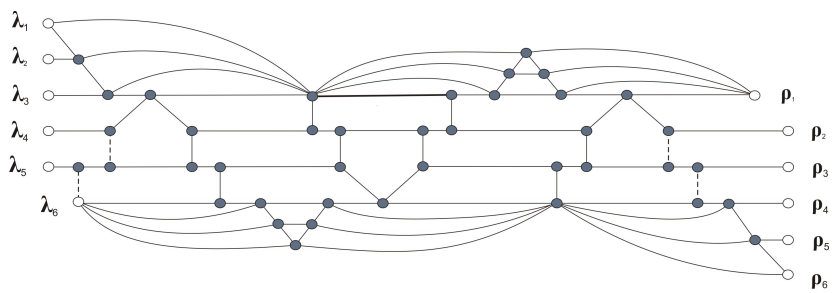

Fig. 1. A tile drawing of the tile $G_{3,4}$. The wall vertices are drawn in white.

The construction defines a graph $G(\ell, n, m)$ with three integer parameters $\ell \geq 1$, $n \geq 3$ and odd $m \geq 3$, as follows. There is a tile $G_{\ell,n}$, with the walls of size $n+\ell-1$, which is illustrated in Figure 1 and formally defined below. Let $\mathcal{G}(\ell, n, m) = (G_{\ell,n}, {}^{\updownarrow}G_{\ell,n}{}^{\updownarrow}, G_{\ell,n} \ldots, {}^{\updownarrow}G_{\ell,n}{}^{\updownarrow}, G_{\ell,n})$ be a sequence of such tiles of length m. The graph $G(\ell, n, m)$ is constructed as $\circ\big(\mathcal{G}(\ell, n, m)^{\updownarrow}\big)$. In the degenerate case of $\ell = 0$, the graph $G(0, n, m)$ is defined as the "staircase strip" graph from Bokal's [3], and $G(0, n, m)$ will be contained in $G(\ell, n, m)$ as a subdivision for every ℓ.

The tile $G_{\ell,n}$ is composed of three copies of a smaller tile $H_{\ell,n}$ such that $G_{\ell,n} = H_{\ell,n} \otimes {}^{\updownarrow}H_{\ell,n}{}^{\updownarrow} \otimes H_{\ell,n}$. A fragment illustrating the join $H_{3,8} \otimes {}^{\updownarrow}H_{3,8}{}^{\updownarrow}$ is presented in Figure 2. Formally, $H_{\ell,n}$ consists of $2\ell + n$ pairwise edge disjoint paths, grouped into three families $P'_1, \ldots, P'_\ell$, $Q'_1, \ldots, Q'_\ell$, and $S'_1, \ldots, S'_n$, and an additional set F' of $2(n-2)$ edges not on these paths.

- The paths $S'_1, \ldots, S'_n$ are pairwise vertex-disjoint except that S'_1 shares one vertex with S'_2 (w_1 in Figure 2). The additional $2(n-2)$ edges of F' are in

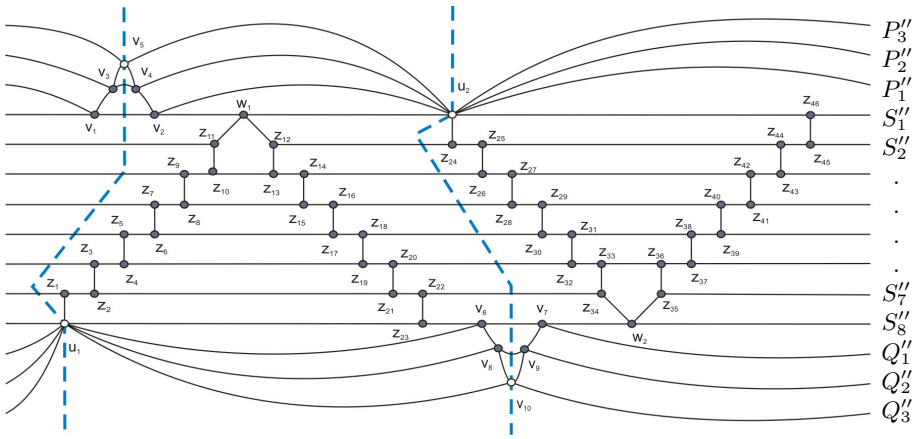

Fig. 2. A fragment of the tile $G_{3,8} = H_{3,8} \otimes^{\updownarrow} H_{3,8}{}^{\updownarrow} \otimes H_{3,8}$; defining the one tile $H_{3,8}$ (left, between the dashed margins) and showing the composition $H_{3,8} \otimes^{\updownarrow} H_{3,8}{}^{\updownarrow}$ in $G_{3,8}$.

pairs between vertices of the paths S'_{i-1} and S'_i for $i = 3, \ldots, n$, as depicted in Figure 2 (edges $u_1z_1, z_2z_3, \ldots, z_{22}z_{23}$).

– The union $S'_1 \cup \ldots \cup S'_n \cup F'$ is (consequently) a subdivision of the aforementioned staircase tile from [3].
– The paths $Q'_1, \ldots, Q'_\ell$ all share the bottom-most vertex u_1 of S'_n on the left wall of $H_{\ell,n}$, and are combined in such a way that Q'_i, $i = 1, \ldots, \ell$, shares exactly one vertex with Q'_{i-1} (with S'_n for $i = 1$) other than u_1 and this shared vertex is of degree 4, as depicted near the right wall in Figure 2 (vertices v_6, v_8, v_{10}). The paths $P'_1, \ldots, P'_\ell$ analogously share the top-most vertex u_2 of S'_1 on the right wall of $H_{\ell,n}$ and are symmetric to Q's.

Let P''_i, Q''_i, S''_i denote the paths obtained as the union of the three copies of each of P'_i, Q'_i, S'_i in $G_{\ell,n}$. Then $P''_1, \ldots, P''_\ell$, $Q''_1, \ldots, Q''_\ell$, and $S''_1, \ldots, S''_n$ are all traversing paths of the tile $G_{\ell,n}$. Let P_i, Q_i, S_i denote the corresponding unions of the paths in whole $G(\ell, n, m)$.

The proof of the following basic properties is straightforward, as attentive reader could easily verify from the illustrating pictures of $H_{\ell,n}$ (recall that degree-2 vertices are removed in a tile join).

Proposition 3.1. *For every $\ell \geq 1$ and $n \geq 3$, the tiles $H_{\ell,n}$, and hence also $G_{\ell,n}$, are perfect planar tiles. The graph $G(\ell, n, m)$ has $3m(2\ell + 4n - 8)$ vertices, out of which $3m \cdot 2\ell$ have degree 4, $3m(4n - 9)$ have degree 3, and remaining $3m$ vertices have degree $2\ell + 3$. The average degree of $G(\ell, n, m)$ is*

$$\frac{5l + 6n - 12}{l + 2n - 4}\,. \qquad \square$$

We conclude with the main desired property of the graph $G(\ell, n, m)$.

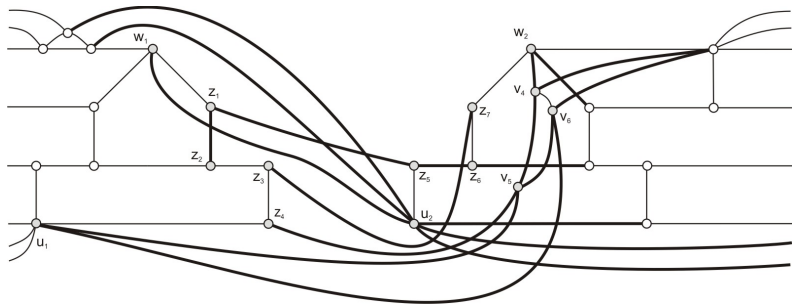

Fig. 3. A fragment of an optimal tile drawing of $G_{2,4}{}^{\updownarrow}$.

Theorem 3.2. *Let $\ell \geq 1$, $n \geq 3$ be integers. Let $k = (\ell^2 + \binom{n}{2} - 1 + 2\ell(n-1))$ and $m \geq 4k - 1$ be odd. Then the graph $G(\ell, n, m)$ is k-crossing-critical.*

Proof. By using Theorem 2.1 and symmetry, it suffices to prove the following:

I) $\mathrm{tcr}\big(\otimes\, \mathcal{G}(\ell, n, m)^{\updownarrow}\big) \geq k$, and
II) every edge of $G_{\ell,n}$ corresponding to one copy of $H_{\ell,n}$ in it is *critical*, meaning that $\mathrm{tcr}(G_{\ell,n}{}^{\updownarrow} - e) < k$ for every edge $e \in E(H_{\ell,n}) \subseteq E(G_{\ell,n})$.

Recall the pairwise edge-disjoint traversing paths $P_1, \ldots, P_\ell$, $Q_1, \ldots, Q_\ell$, and $S_1, \ldots, S_n$ of the composed tile $\otimes\mathcal{G}(\ell, n, m)$. We define the following disjoint sets of pairs of these paths, such that each pair is formed by *vertex-disjoint* paths:

- $\mathcal{A} = \{\{P_i, Q_j\} : 1 \leq i, j \leq \ell\}$ where $|\mathcal{A}| = \ell^2$,
- $\mathcal{B} = \{\{P_i, S_j\} : 1 \leq i \leq \ell, 1 < j \leq n\}$ where $|\mathcal{B}| = \ell(n-1)$,
- $\mathcal{C} = \{\{Q_i, S_j\} : 1 \leq i \leq \ell, 1 \leq j < n\}$ where $|\mathcal{C}| = \ell(n-1)$.

Each pair in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is twisted in $\otimes\mathcal{G}(\ell, n, m)^{\updownarrow}$, and so these pairs account for at least $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| = 2\ell(n-1) + \ell^2$ crossings in a tile drawing of $\otimes\mathcal{G}(\ell, n, m)^{\updownarrow}$, by Lemma 2.2. Importantly, each of these crossings involves at least one edge of $R = P_1 \cup \ldots \cup P_\ell \cup Q_1 \cup \ldots \cup Q_\ell$. The subgraph $\otimes\mathcal{G}(\ell, n, m) - E(R)$ contains a subdivision of the staircase strip $\otimes\mathcal{G}(0, n, m)$. Hence any tile drawing of $\otimes\mathcal{G}(\ell, n, m)^{\updownarrow}$ contains another at least $\mathrm{tcr}\big(\otimes\, \mathcal{G}(0, n, m)^{\updownarrow}\big)$ crossings not involving any edges of R. Since $\mathrm{tcr}\big(\otimes\, \mathcal{G}(0, n, m)^{\updownarrow}\big) \geq \binom{n}{2} - 1$ by [3], we get $\mathrm{tcr}\big(\otimes\, \mathcal{G}(\ell, n, m)^{\updownarrow}\big) \geq \binom{n}{2} - 1 + 2\ell(n-1) + \ell^2 = k$, thus proving (I).

To finish with (II), we investigate the tile drawing in Figure 3. It is routine to count that a natural generalization of this drawing has precisely $\binom{n-1}{2} + (n-2)\ell + (\ell+1)^2 + (\ell+1)(n-3) + \ell = k$ crossings, and so it is optimal. Consequently, every edge which is crossed in Figure 3 is critical, are so are edges which become crossed after suitable local sliding of some vertex or edge (while preserving optimality) in the picture. This way one can easily verify that all the edges of a copy of $H_{2,4}$ in $G_{2,4}$, up to symmetry, are critical; except possibly three z_3z_4, z_5u_2, z_6z_7. The following local changes in the picture verify criticality also for the latter three edges:

– for z_3z_4, slide the edge z_3z_7 up (above u_2) and the edge w_1u_2 slightly down,
– for z_5u_2, z_6z_7, slide the edge z_3z_7 up (above z_6), the edge w_1u_2 down (below z_4), and the edge z_4v_5 together with the vertex v_5 suitably up.

An extension of this argument to the general case of $G_{\ell,n}$ is again routine. □

4 Families with Prescribed Frequent Degrees

In order to fully answer the primary question of this paper—about which vertex degrees other than $3, 4, 6$ can occur arbitrarily often in infinite k-crossing-critical families—we start by repeating the three ingredients we have got so far. First, there is a bunch of established critical constructions essentially covering all the even degree cases and degree 3. Second, we have newly covered the cases of any fixed odd degree in Section 3. And third, we have got the zip product operation.

Proposition 4.1. *There exist (infinite) families $\mathcal{F}$ of simple, 3-connected, k-crossing-critical graphs such that, in addition, the following holds:*

a) ([11, Section 4].) For every $k \geq 10$ or odd $k \geq 5$, and every rational $r \in (4, 6 - \frac{8}{k+1})$, a family $\mathcal{F}$ which is $\{4, 6\}$-max-universal and each member of $\mathcal{F}$ is of average degree exactly r, and another $\mathcal{F}$ which is $\{4\}$-max-universal and of average degree exactly 4. Every graph of the two families has the set of its vertex degrees equal to $\{3, 4, 6\}$ (e.g., degree 3 repeats six times in each).
b) ([11, Section 3 and 4].) For every $\varepsilon > 0$, any integer $k \geq 5$ and every set D_e of even integers such that $\min(D_e) = 4$ and $6 \leq \max(D_e) \leq 2k - 2$, a family $\mathcal{F}$ which is D_e-max-universal, and each graph of $\mathcal{F}$ has the set of its vertex degrees $D_e \cup \{3\}$ and is of average degree from the interval $(4, 4 + \varepsilon)$.
c) ([13] for $k = 2$ and [3] for general k, see $G(0, n, m)$.) For every $k = \binom{n}{2} - 1$ where $n \geq 3$ is an integer, a family $\mathcal{F}$ which is $\{3, 4\}$-max-universal and each member of $\mathcal{F}$ is of average degree equal to $3 + \frac{1}{4n-7}$.
d) ($G(\ell, 3, m)$ in Theorem 3.2.) For every $k = \ell^2 + 4\ell + 2$ where $\ell \geq 1$ is an integer, a family $\mathcal{F}$ which is $\{3, 4, 2\ell + 3\}$-max-universal and each member of $\mathcal{F}$ is of average degree $5 - \frac{4}{\ell+2}$.

Using the zip product and Theorem 2.3, we can hence easily combine all the cases of Proposition 4.1 to obtain the following "ultimate" answer:

Theorem 4.2. *Let D be any finite set of integers such that $\min(D) \geq 3$. Then there is an integer $K = K(D)$, such that for every $k \geq K$, there exists a D-universal family of simple, 3-connected, k-crossing-critical graphs. Moreover, if either $3, 4 \in D$ or both $4 \in D$ and D contains only even numbers, then there exists a D-max-universal such family. All the vertex degrees are from $D \cup \{3, 4, 6\}$.*

We refer to the Appendix for a full formal proof.

5 Families with Prescribed Average Degree

In addition to Theorem 4.2 we are going to show that the claimed D-max-universality property can be combined with nearly any feasible rational average degree of the family. The full statement reads:

Theorem 5.1. *Let D be any finite set of integers such that $\min(D) \geq 3$ and $A \subset \mathbb{R}$ an interval. Assume that at least one of the following assumptions holds:*

a) $D \supseteq \{3,4,6\}$ and $A = (3,6)$,
b) $D \supsetneq \{3,4\}$ and $A = (3,4]$, or $D = \{3,4\}$ and $A = (3,4)$,
c) $D \supsetneq \{3,4\}$ and $A = (3, 5 - \frac{8}{b+1})$ where $b \geq 9$ is the largest odd number in D,
d) $D \supseteq \{4,6\}$ has only even numbers and $A = (4,6)$, or $D = \{4\}$ and $A = \{4\}$.

Then, for every rational $r \in A \cap \mathbb{Q}$, there is an integer $K = K(D,r)$ such that for every $k \geq K$, there exists a D-max-universal family of simple, 3-connected, k-crossing-critical graphs of average degree precisely r.

We refer to the Appendix for a full formal proof of the theorem which we only sketch in this restricted space. The basic idea of balancing the average degree in a crossing-critical family is quite simple; assume we have two families $\mathcal{F}_a, \mathcal{F}_b$ of fixed average degrees $a < b$, respectively, and containing some degree-3 vertices. Then, we can use zip product of graphs from the two families to obtain a new family of average degree equal to a convex combination of a and b. This simple scheme, however, has two difficulties:

I) If one combines graphs $G_1 \in \mathcal{F}_a$ and $G_2 \in \mathcal{F}_b$, the average degree of the disjoint union $G_1 \cup G_2$ is the average of a, b weighted by the sizes of G_1, G_2. Hence we need flexibility in choosing members of $\mathcal{F}_a, \mathcal{F}_b$ of various size.
II) Moreover, after a zip product of G_1, G_2 the resulting average degree is no longer this weighted average of a, b but a slightly different rational number. We take care of this problem by introducing a special *compensation gadget* whose role is to revert the change in average degree caused by zip product.

Addressing (I); a family of graphs $\mathcal{F}$ is *scalable* if all the graphs in $\mathcal{F}$ have equal average degree and for every $G \in \mathcal{F}$ and every integer a, there exists $H \in \mathcal{F}$ such that $|V(H)| = a|V(G)|$. Furthermore, $\mathcal{F}$ is *D-max-universal scalable* if, additionally, H contains at least a vertices of each degree from D and the number of vertices of degrees not in D is bounded independently of a.

Trivially, the families of Proposition 4.1 c),d) are D-max-universal scalable for $D = \{3,4\}$ and $D = \{3,4,2\ell+3\}$, respectively. For families as in Proposition 4.1 a),b), the analogous property can be established by a slight modification of the very flexible construction from [11].

Addressing (II); we again exploit the construction from [11], defining a flexible gadget M_m^c as a special case of Proposition 4.1 a) (see the Appendix for details). The graph M_m^c, for any $m \geq 12$ and $0 \leq c \leq m$, is simple, 3-connected and 5-crossing-critical. The way "compensating by" M_m^c works, is formulated next:

Lemma 5.2. *Let $G_1, \dots, G_t$ be graphs, each having at least two degree-3 vertices, and $q \in \mathbb{N}$. If H is a graph obtained by arbitrarily using the zip product of all $G_1, \dots, G_t$ and of M_m^{q+t}, $m \geq \max(q+t, 12)$, then the average degree of H is equal to the average degree of the disjoint union of $G_1, \dots, G_t$ and M_m^q.*

The next step is to naturally combine available scalable critical families to obtain, with the help of Theorem 2.3 and Lemma 5.2, new families of arbitrary "intermediate" rational average degrees:

Lemma 5.3. *Assume we have simple, D_i-max-universal scalable, 3-connected, k_i-crossing-critical families $\mathcal{F}_i$ of average degree r_i, $i = 1, \dots, t$, such that $r_1 < r_2$. Then for every $k \geq k_1 + \cdots + k_t + 5$ and any $r \in (r_1, r_2) \cap \mathbb{Q}$, there exists a $(D_1 \cup \cdots \cup D_t)$-max-universal family of simple, 3-connected, k-crossing-critical graphs of average degree exactly r.*

While leaving technical details of these tools to the Appendix, we finish with an overview of their case-specific application to Theorem 5.1:

Proof (of Theorem 5.1). The case d) has already been proved in [11], see Proposition 4.1 a). In all other cases, let $\mathcal{F}_1$ be the family from Proposition 4.1 c) such that the parameter n satisfies $r_1 = 3 + \frac{1}{4n-7} < r$ (where $r \in A \cap \mathbb{Q}$, $r > 3$, is the desired fixed average degree).

In the case a), let $\mathcal{F}_2$ be a family from Proposition 4.1 a) with average degree equal to arbitrary (but fixed) $r_2 \in (r, 6) \neq \emptyset$, and chosen as scalable. In the case c), let $\mathcal{F}_2$ be the family from Proposition 4.1 d) for the parameter ℓ such that $b = 2\ell + 3$; in this case $r_2 = 5 - \frac{8}{b+1} > r$. Finally, consider the remaining sub-cases of b). If $D = \{3, 4\}$, then let $\mathcal{F}_2$ be the second family from Proposition 4.1 a) with average degree $r_2 = 4$. If $D \supsetneq \{3, 4\}$, then let $\mathcal{F}_2$ be the family from Proposition 4.1 b), made scalable and of fixed average degree $r_2 > 4$.

In each of the choices of $\mathcal{F}_1, \mathcal{F}_2$ above, it holds $r_1 < r < r_2$. Furthermore, if needed to fulfil D-max-universality, add more scalable families $\mathcal{F}_3, \dots$ as in the proof of Theorem 4.2. Theorem 5.1 then follows directly from Lemma 5.3. □

6 Final Remarks

In the previous constructions, we have always assumed that the fixed crossing number k of the families is sufficiently large. One can, on the other hand, ask what happens if we fix a small value of k beforehand (i.e., independently of the asked degree properties).

In this direction, there is the remarkable result of Dvořák and Mohar [6] proving the existence of k-crossing-critical families with unbounded maximum degree for any $k \geq 171$. Unfortunately, since [6] is not really constructive, we do not know anything exact about the degrees occurring in these families. An explicit construction of a k-crossing-critical family with unbounded maximum degree is known only in the projective plane [12] for $k \geq 2$, but that falls outside of the area of interest of this paper.

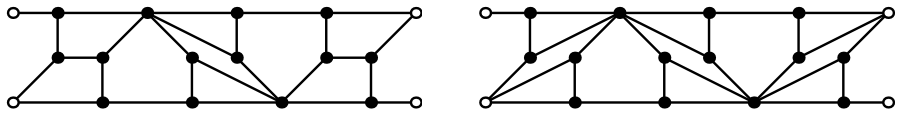

Fig. 4. Fractions (each of three tiles) of constructions of simple, 3-connected, 2-crossing-critical and D-max-universal families for $D = \{3, 5\}$ (left) and $D = \{3, 6\}$ (right).

It thus appears natural to thoroughly investigate the least non-trivial case of $k = 2$, with help of the remarkably involved characterization result [5][3]. Due to limited space, we can only very briefly survey the obtained results, and refer to the Appendix for extensive details and the proofs.

Theorem 6.1. *A simple, 3-connected 2-crossing-critical D-max-universal family exists if and only if $\{3\} \subsetneq D \subseteq \{3, 4, 5, 6\}$. Without the simplicity requirement, such a family exists if and only if $D \subseteq \{3, 4, 5, 6\}$, $|D| \geq 2$, and $D \cap \{3, 4\} \neq \emptyset$.*

We remark that it is important that Theorem 6.1 deals with infinite such families (via the universality property) since not all of the (finitely many) sporadic small 2-crossing-critical graphs are explicitly known [5]. Examples of two sub-cases of Theorem 6.1 can be found in Figure 4.

Theorem 6.2. *A simple, 3-connected, 2-crossing-critical infinite family of graphs with average degree $r \in \mathbb{Q}$ exists if and only if $r \in [3\frac{1}{5}, 4]$. Without the simplicity requirement, such a family exists if and only if $r \in [3\frac{1}{5}, 4\frac{2}{3}]$.*

At last, we return to the statement of Theorem 4.2, which always requires $4 \in D$. On the other hand, from Theorem 6.1 we know that there exist D-max-universal families of simple, 3-connected, 2-crossing-critical graphs for $D = \{3, 5\}$ and $D = \{3, 6\}$ (Figure 4), e.g., when $4 \notin D$, and these can be generalized to any $k > 2$ by a zip product with copies of $K_{3,3}$.

Hence it is an interesting open question of whether there exists a D-max-universal k-crossing-critical family such that $D \cap \{3, 4\} = \emptyset$. It is unlikely that the answer would be easy since the question is related to another long standing open problem—whether there exists a 5-regular k-crossing-critical infinite family. Related to this is the same question of existence of a 4-regular family, which does exist for $k = 3$ [17] and the construction can be generalized to any $k \geq 6$, but the cases $k = 4, 5$ remain open.

Many more questions can be asked in a direct relation to the statement of Theorem 5.1, but we can mention only a few of the interesting ones. E.g., if $6 \notin D$, can the average degree of such a family be from the interval $[5, 6)$? Or, assuming $3 \in D$ but $4 \notin D$, for which sets D one can achieve D-max-universality and what are the related average degrees?

We finish with another interesting structural conjecture:

[3] Even though this very long manuscript [5] is not published yet, its main result has been known already for many years and it is widely believed to be right.

Conjecture 6.3. There is a function $g : \mathbb{N} \to \mathbb{R}^+$ such that, any simple 3-connected k-crossing-critical graph has average degree greater than $3 + g(k)$.

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A Appendix

In this appendix we provide extended full versions with complete proofs of the sections which had to be shortened due to space restrictions in the main paper. (We also repeat all the formal statements.)

A-2 Tools for constructing crossing-critical graphs

A principal tool used in construction of crossing-critical graphs are tiles. They were used already in the early papers on infinite families of crossing-critical graphs by Kochol [13] and Richter and Thomassen [17], although they were formalized only in the work of Pinnontoan and Richter [15, 16], answering Salazar's question [18] on average degrees in infinite families of k-crossing-critical graphs. Bokal built upon these results to fully settle Salazar's question when combining tiles with zip product [3]. Also a recent result that all large 2-crossing-critical graphs are composed of large multi-sets of 42 different tiles [5] demonstrates that tiles are intimately related to crossing-critical graphs. In this section, we summarize the known results from [3, 5, 15], which we need for our constructions.

Tiles are essentially graphs equipped with two sequences of vertices that are identified among tiles or within a tile in order to, respectively, form new tiles or graphs of tiles. The tiles can be drawn in unit square respecting the order of special sequences of vertices, thus providing special, restricted drawings of tiles. Due to the restriction, the crossing number of these special drawings is an upper bound to the crossing number of either underlying graphs, or the graphs obtained by identifying these specific vertices. The formal concepts allowing these operations are summarized in the following definition and the lemma immediately after it:

Definition A-2.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\rho = (\rho_1, \ldots, \rho_r)$ be two sequences of distinct vertices of a graph G, where no vertex of G appears in both λ and ρ.*

1. *For any sequence λ, let $\bar{\lambda}$ denote its reversed sequence.*
2. *A* tile *is a triple $T = (G, \lambda, \rho)$.*
3. *The sequence of vertices λ is called the* left wall *and the sequence of vertices ρ is called the* right wall *of T.*
4. *A* tile drawing *of a tile $T = (G, \lambda, \rho)$ is a drawing of G in unit square $[0, 1] \times [0, 1]$ such that:*
   - *all vertices of the left wall are drawn in $\{0\} \times [0, 1]$ and all vertices of the right wall are drawn in $\{1\} \times [0, 1]$;*
   - *y-coordinates of wall vertices consistently respect the order in sequences λ and ρ.*
5. *The* tile crossing number $\mathrm{cr}(T)$ *of a tile T is the smallest crossing number over all tile drawings of T.*
6. *A tile $T = (G, \lambda, \rho)$ is* compatible *with a tile $T' = (G', \lambda', \rho')$ if $|\rho| = |\lambda'|$ and* cyclically-compatible *if it is compatible with itself.*
7. *A sequence of tiles $(T_0, \ldots, T_m)$ is compatible, if tiles T_i and T_{i+1} are compatible for $i = 0, \ldots, m - 1$.*

8. *A sequence is* cyclically-compatible *if also T_m is compatible with T_0.*
9. *The join of two compatible tiles $T = (G, \lambda, \rho)$ and $T' = (G', \lambda', \rho')$ is defined as the tile $T \otimes T' = (G \otimes G', \lambda, \rho')$, where $G \otimes G'$ represents the graph obtained from the union of graphs G and G', by identifying ρ_i with λ'_i for $i = 1, \ldots, |\rho|$. We can get vertices of degree two by joining two tiles. We remove these vertices and contract an incident edge. By joining two tiles, we can also get double edges, and we keep them.*
10. *Since the operator $\otimes$ is associative, we can define the join of a compatible sequence of tiles $\mathcal{T} = (T_0, \ldots, T_m)$ as $\otimes\mathcal{T} = T_0 \otimes \ldots \otimes T_m$.*
11. *Let $T = (G, \lambda, \rho)$ be a cyclically-compatible tile. The* cyclization *$\circ T$ of a tile T is the graph G in which we identify λ_i with ρ_i for $i = 1, \ldots, |\lambda|$.*
12. *Let $T_0 = (G, \lambda, \rho)$, …, $T_m = (G', \lambda', \rho')$ be cyclically-compatible tiles. The* cyclization *of a cyclically-compatible sequence of tiles is defined as $\circ\mathcal{T} = \circ(T_0 \otimes \ldots \otimes T_m)$, where we identify ρ'_i with λ_i for $i = 1, \ldots, |\lambda|$.*
13. *Let $T = (G, \lambda, \rho)$ be a tile. The* right-inverted *tile $T^{\updownarrow}$ is the tile $(G, \lambda, \bar{\rho})$ and the* left-inverted *tile ${}^{\updownarrow}T$ is the tile $(G, \bar{\lambda}, \rho)$. The* inverted *tile of T is the tile ${}^{\updownarrow}T^{\updownarrow} = (G, \bar{\lambda}, \bar{\rho})$ and the* reversed *tile of T is the tile $T^{\leftrightarrow} = (G, \rho, \lambda)$.*
14. *For a compatible sequence of tiles $\mathcal{T}$, we define a* twist *as the sequence $\mathcal{T}^{\updownarrow} = (T_0, \ldots, T_m^{\updownarrow})$, and we define an i-*cut *of $\mathcal{T}$ as the sequence $\mathcal{T}/i = (T_{i+1}, \ldots, T_m, T_0, \ldots, T_{i-1})$.*

Lemma A-2.2 ([15]). *Let T be a cyclically-compatible tile. Then, $\mathrm{cr}(\circ T) \leq \mathrm{tcr}(T)$. Let $\mathcal{T} = (T_0, \ldots, T_m)$ be a compatible sequence of tiles. Then, $\mathrm{tcr}(\otimes\mathcal{T}) \leq \sum_{i=0}^{m} \mathrm{tcr}(T_i)$.*

The above Lemma applies without any information on the internal structure of the tiles. However, by exploiting their internal structure (planarity and enough connectivity), we can also prove a lower bound on the tile crossing number, which can, with sufficiently many tiles, be exploited for the lower bound on the crossing number of the graph resulting from the tile. Prerequisites for these applications are summarized in the following definition and applied in the theorem that follows.

Definition A-2.3. *Let $T = (G, \lambda, \rho)$ be a tile. Then:*

1. *T is* connected *if G is connected.*
2. *T is* planar *if $\mathrm{tcr}(T) = 0$.*
3. *T is* perfect *if the following holds:*
   - *$|\lambda| = |\rho|$;*
   - *$G - \lambda$ and $G - \rho$ are connected;*
   - *for every $v \in \lambda$ there is a path to the right wall ρ in G internally disjoint from λ and for every $u \in \rho$ there is a path to the left wall λ in G internally disjoint from ρ;*
   - *for every $0 \leq i < j \leq |\lambda|$, there is a pair of disjoint paths, one joining λ_i and ρ_i, and the other joining λ_j and ρ_j.*

Theorem A-2.4 ([3]). *Let $\mathcal{T} = (T_0, \ldots, T_l, \ldots, T_m)$ be a cyclically-compatible sequence of tiles. Assume that, for some integer $k \geq 0$, the following hold: $m \geq$*

$4k-2$, $\mathrm{tcr}(\otimes\mathcal{T})/i) \geq k$, *and the tile* T_i *is a perfect planar tile, both for every* $i = 0, \ldots, m$, $m \neq l$. *Then,* $\mathrm{cr}(\circ\mathcal{T}) \geq k$.

This theorem can yield exact crossing number under the assumptions of the next corollary.

Corollary A-2.5 ([3]). *Let* $\mathcal{T} = (T_0, \ldots, T_l, \ldots, T_m)$ *be a cyclically-compatible sequence of tiles and* $k = \min_{i \neq l} \mathrm{tcr}(\otimes(\mathcal{T}/i))$. *If* $m \geq 4k-2$ *and the tile* T_i *is a perfect planar tile for every* $i = 0, \ldots, m$, $i \neq l$, *then* $\mathrm{cr}(\circ\mathcal{T}) = k$.

Exact lower bounds facilitate establishing criticality of the tiles and graphs, as the smallest drop in crossing number suffices for criticality of an edge. For combinatorially handling the criticality of the constructed graph on the basis of the properties of tiles, we introduce *degeneracy* of tiles and criticality of sequences of tiles as follows:

Definition A-2.6. *1. A tile* T *is* k*-degenerate if it is planar and* $\mathrm{tcr}(T^{\updownarrow} - e) < k$ *for any* $e \in E(T)$.
2. A sequence $\mathcal{T} = (T_0, \ldots, T_m)$ *is* k*-critical if the tile* T_i *is* k*-degenerate for every* $i = 0, \ldots, m$ *and* $\min_{i \neq m} \mathrm{tcr}(\otimes(\mathcal{T}^{\updownarrow}/i)) \geq k$.

Using these concepts, Corollary A-2.5 can be applied to establish criticality of graphs resulting from crossing critical sequences of tiles or from degenerate tiles.

Corollary A-2.7 ([3]). *Let* $\mathcal{T} = (T_0, \ldots, T_m)$ *be a* k*-critical sequence of tiles. Then,* $T = \otimes\mathcal{T}$ *is a* k*-degenerate tile. If* $m \geq 4k-2$ *and* $\mathcal{T}$ *is cyclically-compatible, then* $\circ(T^{\updownarrow})$ *is a* k*-crossing-critical graph.*

To estimate the tile crossing number, we use an informal tool called *gadget*. This can be any structure inside of a tile T, which guarantees a certain number of crossings in every tile drawing of T. The gadgets we use are twisted pairs of paths, guaranteeing one crossing each, and staircase strips of width n, guaranteeing $\binom{n}{2} - 1$ crossings.

Definition A-2.8. *A* traversing path *in a tile* $T = (G, \lambda, \rho)$ *is a path* P *in a graph* G, *for which there exist indices* $i(P) \in \{1, \ldots, |\lambda|\}$, *and* $j(P) \in \{0, \ldots, |\rho|\}$, *so that* P *is a path from* $\lambda(P) = \lambda_{i(P)}$ *to* $\rho(P) = \rho_{j(P)}$ *and* $\lambda(P)$ *and* $\rho(P)$ *are the only wall vertices of* P.

A pair of disjoint traversing paths $\{P, Q\}$ *is* twisted *if* $i(P) < i(Q)$ *and* $j(P) > j(Q)$, *and* aligned *otherwise. A family* $\mathcal{F}$ *of pairs of disjoint traversing paths is* aligned, *if all the pairs in* $\mathcal{F}$ *are aligned. The family is* twisted, *if all the pairs are twisted.*

If the traversing paths in a twisted pair $\{P, Q\}$ are disjoint, this implies one crossing in any tile drawing of T. This is generalized to twisted families in the following lemma:

Lemma A-2.9 ([3]). *Let* $\mathcal{F}$ *be a twisted family in a tile* T, *such that no edge occurs in two distinct paths of* $\cup\mathcal{F}$. *Then,* $\mathrm{tcr}(T) \geq |\mathcal{F}|$.

A staircase sequence of width n, which is a cyclically-compatible sequence of specific tiles of odd length, was introduced in [3]. The definition is rather technical and we do not repeat it here; it suffices to know that the paths S_i spanned in the tile $\mathcal{G}(\ell, n, m)$ in Section 3 induce a staircase strip, to which the following Theorem applies:

Theorem A-2.10 ([3]). *Let T be a tile and assume that $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ forms a twisted staircase strip of width n in T. Then,* $\mathrm{tcr}(T) \geq \binom{n}{2} - 1$.

This concludes our discussion of known results on tiles in graphs. Tiled graphs are joined together using zip product construction [2, 3]. We use the version restricted to vertices of degree three, as introduced in [11].

Definition A-2.11. *For $i \in \{1, 2\}$, let G_i be a simple graph and let $v_i \in V(G_i)$ be its vertex of degree 3, such that $G_i - v_i$ is connected. We denote the neighbours of v_i by u^i_j for $j \in \{1, 2, 3\}$. The zip product of G_1 and G_2 according to vertices v_1, v_2 and their neighbours, is obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding three edges $u^1_1u^2_1$, $u^1_2u^2_2$, $u^1_3u^2_3$.*

While crossing number is super-additive over general zip products only under a technical connectivity condition, the following theorem holds for zip products of degree (at most) three:

Theorem A-2.12 ([4]). *Let G be a zip product of G_1 and G_2 as in Definition A-2.11. Then,* $\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$. *Consequently, if G_i is k_i-crossing-critical for $i = 1, 2$, then G is $(k_1 + k_2)$-crossing-critical.*

A-4 Families with Prescribed Frequent Degrees

We now get back to the primary question which motivated the research leading to [11] and this paper: which vertex degrees other than $3, 4, 6$ can occur arbitrarily often in infinite k-crossing-critical families? First, we summarize the relevant particular constructions—our future building blocks—obtained so far (note that some of the claimed results have been proved in a more general form than stated here, but we state them right in the form we shall use).

Proposition A-4.1. *There exist (infinite) families $\mathcal{F}$ of simple, 3-connected, k-crossing-critical graphs such that, in addition, the following holds:*

a) ([11, Section 4].) For every $k \geq 10$ or odd $k \geq 5$, and every rational $r \in (4, 6 - \frac{8}{k+1})$, a family $\mathcal{F}$ which is $\{4, 6\}$-max-universal and each member of $\mathcal{F}$ is of average degree exactly r, and another $\mathcal{F}$ which is $\{4\}$-max-universal and of average degree exactly 4. Every graph of the two families has the set of its vertex degrees equal to $\{3, 4, 6\}$ (e.g., degree 3 repeats six times in each).

b) ([11, Section 3 and 4].) For every $\varepsilon > 0$, any integer $k \geq 5$ and every set D_e of even integers such that $\min(D_e) = 4$ and $6 \leq \max(D_e) \leq 2k - 2$, a family $\mathcal{F}$ which is D_e-max-universal, and each graph of $\mathcal{F}$ has the set of its vertex degrees $D_e \cup \{3\}$ and is of average degree from the interval $(4, 4 + \varepsilon)$.

c) *([13] for $k = 2$ and [3] for general k, see $G(0, n, m)$.) For every $k = \binom{n}{2} - 1$ where $n \geq 3$ is an integer, a family $\mathcal{F}$ which is $\{3, 4\}$-max-universal and each member of $\mathcal{F}$ is of average degree equal to $3 + \frac{1}{4n-7}$.*

d) *($G(\ell, 3, m)$ in Theorem 3.2.) For every $k = \ell^2 + 4\ell + 2$ where $\ell \geq 1$ is an integer, a family $\mathcal{F}$ which is $\{3, 4, 2\ell + 3\}$-max-universal and each member of $\mathcal{F}$ is of average degree $5 - \frac{4}{\ell+2}$.*

Having the particular constructions of Proposition A-4.1 and the zip product with Theorem A-2.12 at hand, it is now quite easy to give the "ultimate" combined construction as follows. For two graph families $\mathcal{F}_1, \mathcal{F}_2$ of simple 2-connected graphs such that each graph in $\mathcal{F}_1 \cup \mathcal{F}_2$ has a vertex of degree 3, we define the *zip product* of $\mathcal{F}_1$ and $\mathcal{F}_2$ as the family of all graphs H such that there exist $G_1 \in \mathcal{F}_1$, $G_2 \in \mathcal{F}_2$ and vertices $v_1 \in V(G_1)$, $v_2 \in V(G_2)$ of degree 3, and H is the zip product of G_1 and G_2 according to v_1, v_2.

Lemma A-4.2. *Let $\mathcal{F}_i$, $i = 1, 2$, be a D_i-max-universal family of simple 2-connected graphs such that each graph in $\mathcal{F}_i$ has a vertex of degree 3. Then the zip product of $\mathcal{F}_1$ and $\mathcal{F}_2$ is a $(D_1 \cup D_2)$-max-universal family.*

Proof. Let $\mathcal{F}$ denote the zip product of $\mathcal{F}_1$ and $\mathcal{F}_2$. We first prove that $\mathcal{F}$ is $(D_1 \cup D_2)$-universal. Choose any set of integers $\{m_d \mid d \in D_1 \cup D_2\}$, and graphs $G_i \in \mathcal{F}_i$, $i = 1, 2$, such that G_i contains at least m_d vertices of degree d for each $d \in D_i$, and G_i has at least $m_3 + 1$ vertices of degree 3 if $3 \in D_i$. Then the zip product of G_1 and G_2 (according to any pair of their degree-3 vertices) has at least m_d vertices of degree d for each $d \in D_1 \cup D_2$.

Conversely, assume that $\mathcal{F}$ is $\{d\}$-universal for some integer d. Then for every integer m there exists $G \in \mathcal{F}$ such that G has at least $2m$ vertices of degree d. Since G is a zip product of graphs $G_i \in \mathcal{F}_i$, $i = 1, 2$, one of G_1, G_2 contains at least m vertices of degree d. W.l.o.g., this happens infinitely often for $i = 1$, and so (up to symmetry) $\mathcal{F}_1$ is $\{d\}$-universal. Therefore, $d \in D_1 \cup D_2$ which proves that $\mathcal{F}$ is $(D_1 \cup D_2)$-max-universal. □

Theorem A-4.3 (Theorem 4.2). *Let D be any finite set of integers such that $\min(D) \geq 3$. Then there is an integer $K = K(D)$, such that for every $k \geq K$, there exists a D-universal family of simple, 3-connected, k-crossing-critical graphs. Moreover, if either $3, 4 \in D$ or both $4 \in D$ and D contains only even numbers, then there exists a D-max-universal such family. All the vertex degrees in the families are from $D \cup \{3, 4, 6\}$.*

Proof. It suffices to prove the second claim (D-max-universal) since a $(D \cup \{3, 4\})$-max-universal family is also D-universal. Furthermore, if D contains only even numbers, then the claim has already been proved in [11], here in Proposition A-4.1 b).

Hence assume the case $3, 4 \in D$, and let $D_e \subseteq D$ be the subset of the even integers from D. Let $\mathcal{F}_e$ denote the family from Proposition A-4.1 b) for $k_e = \frac{1}{2}\max(D_e) + 1$, and $\mathcal{F}_3$ the family from Proposition A-4.1 c) for $k = 2$. For every $a \in D \setminus D_e$, $a > 3$, let $\mathcal{F}_a$ denote the family from Proposition A-4.1 d)

for $2\ell_a+3=a$ and the crossing number $k_a=\ell_a^2+4\ell_a+2$. Since, in particular, $\mathcal{F}_3$ is $\{3\}$-universal, we may assume that every graph in $\mathcal{F}_3$ has more than $|D\setminus D_e|$ vertices of degree 3. We now construct a family $\mathcal{F}$ as the iterated zip product of $\mathcal{F}_3$, $\mathcal{F}_e$, and (possibly) of each $\mathcal{F}_a$ where $a\in D\setminus D_e$, $a>3$.

Clearly, every graph from $\mathcal{F}$ is simple 3-connected. By Lemma A-4.2, $\mathcal{F}$ is moreover D-max-universal, and by Theorem A-2.12, $\mathcal{F}$ is K-crossing-critical where $K=k_e+2+\sum_{a\in D\setminus D_e, a>3}k_a$. This construction creates only vertices of degrees from $D\cup\{3,4,6\}$. To extend the construction of $\mathcal{F}$ to any parameter $k>K$, we simply replace the family $\mathcal{F}_e$ by analogous $\mathcal{F}'_e$ from Proposition A-4.1 b) with the parameter $k'_e=k_e+(k-K)$. □

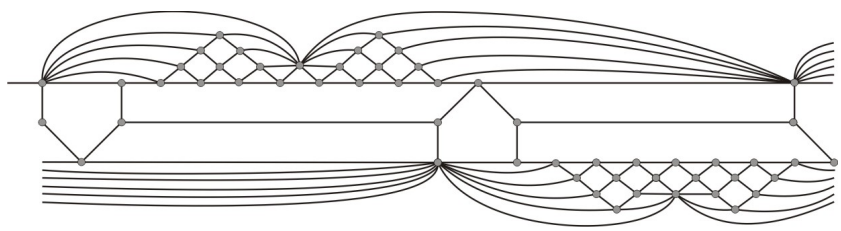


Fig. 5. A possible way of combining the ideas of the construction [11] with the tile $G_{5,3}$.

At last we shortly remark that building blocks of the "crossed belt" construction of [11] (Proposition A-4.1 b) can be directly combined with the new construction of $G(\ell,n,m)$, without invoking a zip product. Such a combination is outlined in Figure 5. However, since this construction can only achieve a combination of various even degrees with one prescribed odd degree (greater than 3), it cannot replace the proof of Theorem A-4.3 and so we refrain from giving the lengthy technical details in this paper.

A-5 Families with Prescribed Average Degree

In addition to Theorem A-4.3 we are going to show that the claimed D-max-universality property can be combined with nearly any feasible rational average degree of the family.

Theorem A-5.1 (Theorem 5.1). *Let D be any finite set of integers such that $\min(D)\geq 3$ and $A\subset\mathbb{R}$ be an interval of reals. Assume that at least one of the following assumptions holds:*

a) $D\supseteq\{3,4,6\}$ and $A=(3,6)$,
b) $D\supsetneq\{3,4\}$ and $A=(3,4]$, or $D=\{3,4\}$ and $A=(3,4)$,
c) $D\supsetneq\{3,4\}$ and $A=(3,5-\frac{8}{b+1})$ where b is the largest odd number in D and $b\geq 9$,

d) $D \supseteq \{4,6\}$ contains only even numbers and $A = (4,6)$, or $D = \{4\}$ and $A = \{4\}$.

Then, for every rational $r \in A \cap \mathbb{Q}$, there is an integer $K = K(D,r)$ such that for every $k \geq K$, there exists a D-max-universal family of simple 3-connected k-crossing-critical graphs of average degree precisely r.

Before we prove the theorem, we informally review the coming steps. The basic idea of balancing the average degree in a crossing-critical family is quite simple; assume we have two families $\mathcal{F}_a, \mathcal{F}_b$ of fixed average degrees $a < b$, respectively, and containing some degree-3 vertices. Then, we can use zip product of graphs from the two families to obtain new graphs of average degrees which are convex combinations of a and b. This simple scheme, however, has two difficulties:

- If one combines graphs $G_1 \in \mathcal{F}_a$ and $G_2 \in \mathcal{F}_b$, then the average degree of the disjoint union $G_1 \cup G_2$ is the average of a, b weighted by the sizes of G_1, G_2. Hence we need great flexibility in choosing members of $\mathcal{F}_a, \mathcal{F}_b$ of various size, and this will be taken care of by the notion of a *scalable family*.
- Second, after applying a zip product of G_1, G_2 the resulting average degree is no longer this weighted average of a, b but a slightly different rational number. We will take care of this problem by introducing a special *compensation gadget* whose role is to revert the change in average degree caused by zip product.

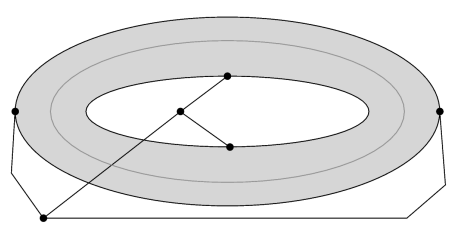


Fig. 6. The k-crossing-critical "crossed belt" construction of [11]: the shaded part is any plane graph consisting of an edge-disjoint union of k cycles, satisfying certain (rather weak) technical and connectivity conditions; the six marked vertices are all of degree three.

We start with addressing the second point. The *compensation gadget* (one for a whole family) will be picked from the family in Proposition A-4.1 a). To describe it precisely, we have to (at least informally) introduce the very general *crossed belt* construction of crossing-critical families from [11]—see it is Figure 6. Let T be the planar tile depicted in Figure 7 on the left, and let M'_m be the planar graph obtained as the cyclization $\circ(T_0, \ldots, T_{m-1})$ where each $T_i = T$. Let M_m, $m \geq 12$, be constructed from M'_m by adding six new degree-3 vertices and five

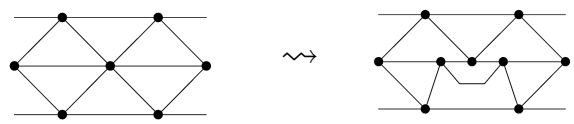


Fig. 7. The tile T (left) used to construct our "compensation gadget" M_m, and the tile T'' (called "double-split" in [11]) that can replace T in the compensation gadget.

new edges as in Figure 6, such that four of the new vertices subdivide rim edges of the tiles $T_0, T_{\lfloor m/4 \rfloor}, T_{\lfloor m/2 \rfloor}, T_{\lfloor 3m/4 \rfloor}$. Let M_m^c be constructed exactly as M_m but replacing arbitrary $c \geq 0$ of the tiles T with T'' shown on the right in Figure 7.

Proposition A-5.2 ([11]). *The graph M_m^c, for any $m \geq 12$ and $0 \leq c \leq m$, is 5-crossing-critical.*

The way "compensating by" the gadget M_m^c works, is formulated next.

Lemma A-5.3 (Lemma 5.2). *Let $G_1, \ldots, G_t$ be graphs, each having at least two degree-3 vertices, and $q \in \mathbb{N}$. If H is a graph obtained using the zip product of all $G_1, \ldots, G_t$ and of M_m^{q+t} (in any order and any way, and for any $m \geq \max(q+t, 12)$), then the average degree of H is equal to the average degree of the disjoint union of $G_1, \ldots, G_t$ and M_m^q.*

Proof. Let $n_i = |V(G_i)|$ and s_i be the sum of degrees in G_i, and let $n_0 = 6m + 6 + 2q$, $s_0 = 28m + 18 + 6q$ be the same quantities in M_m^q. Then $n_0'' = |V(M_m^{q+t})| = n_0 + 2t$ and the sum of degrees of M_m^{q+t} is $s_0'' = s_0 + 6t$. Since performing one zip operation decreases the number of vertices by 2 and the sum of degrees by 6, we have $|V(H)| = n_0'' + n_1 + \cdots + n_t - 2t = n_0 + n_1 + \cdots + n_t$ and the sum of degrees in H is $s_0'' + s_1 + \cdots + s_t - 6t = s_0 + s_1 + \cdots + s_t$, and the claim follows. □

To address the first point, we give the following definition. A family of graphs $\mathcal{F}$ is *scalable* if all the graphs in $\mathcal{F}$ have equal average degree and for every $G \in \mathcal{F}$ and every integer a, there exists $H \in \mathcal{F}$ such that $|V(H)| = a|V(G)|$. Furthermore, $\mathcal{F}$ is *D-max-universal scalable* if, additionally, H contains at least a vertices of each degree from D and the number of vertices of degrees not in D is bounded from above independently of a.

Trivially, the families of Proposition A-4.1 c),d) are D-max-universal scalable for $D = \{3, 4\}$ and $D = \{3, 4, 2\ell + 3\}$, respectively. For the families as in Proposition A-4.1 a),b), we have:

Lemma A-5.4. *There exist families, satisfying the conditions of Proposition A-4.1 a),b), which are D-max-universal scalable for their respective sets D.*

Note that in the case extending Proposition A-4.1 b), the newly constructed family will also have fixed average degree.

Proof. The proof is completely based on the constructions from [11], but since the question of scalability is not considered there, we have to discuss some further details of the crossed belt construction of [11] (recall Figure 6).

First, consider a $\{4\}$-max-universal family $\mathcal{F}_4$ of simple, 3-connected, k-crossing-critical graphs of average degree 4, as in Proposition A-4.1 a). Pick any $G \in \mathcal{F}_4$; then G has precisely six degree-3 vertices, and since the only other vertex degrees occurring in G are 4 and 6, G has precisely three degree-6 vertices. Let G' be the "planar belt" of G (the shaded part in Figure 6, without degree-3 vertices). Then G' can be cut to form a perfect planar tile $T_{G'}$ such that $\circ T_{G'} = G'$. For integer $a \geq 1$, let G'_a denote the cyclization of a copies of $T_{G'}$, and let G''_a denote the graph G'_a with the six degree-3 vertices added back (such that four of them subdivide the same edges of one copy of the tile $T_{G'}$ as they do in G). By [11], G''_a is again k-crossing-critical. If $n = |V(G')|$ and s is the degree sum of G', then $|V(G)| = n + 6$ and the degree sum of G is $s + 18$. Furthermore, $|V(G''_a)| = an + 6$ and the degree sum of G''_a is $as + 18$, and G''_a has $3a$ degree-6 vertices. We denote by G_a the graph obtained by $3a - 3$ "double split" operations each replacing a degree-6 vertex by three degree-4 vertices as illustrated in Figure 7. Then $|V(G_a)| = an + 6 + 2(3a - 3) = a|V(G)|$ and the degree sum of G_a is $as + 18 + 6(3a - 3) = a(s + 18)$, and so the average degree is the same as of G. There are only three degree-6 vertices left in G_a. Hence we may assume $G_a \in \mathcal{F}_4$ as well, for every $a > 1$.

Second, consider a $\{4, 6\}$-max-universal family $\mathcal{F}_r$ of simple 3-connected k-crossing-critical graphs of average degree $r \in (4, 6 - \frac{8}{k+1})$, as in Proposition A-4.1 a). Then the proof follows the same line as in the previous paragraph, only that now we have many degree-6 vertices by the assumption of $\{6\}$-universality.

Third, consider a D_e-max-universal family $\mathcal{F}_e$ of simple 3-connected k-crossing-critical graphs, as in Proposition A-4.1 b). This case is somehow different from the previous two since we have no vertices of degree 6 (unless $6 \in D_e$) and $\mathcal{F}_e$ contains graphs of various average degrees. Though, $\mathcal{F}_e$ can be chosen such that the average degree of every member of $\mathcal{F}_e$ is from the interval $(4, 4 + \varepsilon/2)$ for any fixed $\varepsilon > 0$. Pick arbitrary but sufficiently large $G \in \mathcal{F}_e$. Then one can find (see [11] for details) three edges in G not close to each other and not having vertices of degree other than 4 in close neighbourhood, and let G_1 be obtained by contracting these three edges (into vertices of degree 6). By [11], G_1 is again k-crossing-critical. Since G is sufficiently large, the average degree of G_1 is equal to some $r_1 \in (4, 4 + \varepsilon)$. Now the construction from the first case above applies to G_1 and gives a whole scalable family of average degree r_1. □

The next step is to combine suitable scalable families to obtain arbitrary rational average degrees in a given interval (roughly, between the sparsest and the densest available family).

Lemma A-5.5. *Assume, for $i = 1, \ldots, t$, that $\mathcal{F}_i$ is a D_i-max-universal scalable family of simple 3-connected k_i-crossing-critical graphs of average degree exactly r_i, and that every graph in $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_t$ has at least two degree-3 ver-*

tices. For every $k \geq k_1 + \cdots + k_t + 5$ there exists rational $r_0 \in (3, 6)$ such that, for every $a_1, \ldots, a_t, c \in \mathbb{N}$, the following holds:

a) there exists a simple, 3-connected, k-crossing-critical graph G having at least a_i vertices of each degree from D_i,
b) the number of vertices of G of degree not in $D_1 \cup \cdots \cup D_t$ is bounded from above by a number depending only on c, k and the families $\mathcal{F}_1, \ldots, \mathcal{F}_t$, and
c) the average degree of G is precisely

$$r = \frac{\sum_{i=1}^{t} a_i r_i + c r_0}{\sum_{i=1}^{t} a_i + c}\,. \tag{1}$$

Proof. Let $\ell = k - (k_1 + \cdots + k_t + 5)$ and denote by $\mathcal{K}_\ell$ a set of ℓ disjoint copies of the graph $K_{3,3}$. Pick arbitrary $G_i \in \mathcal{F}_i$, $i = 1, \ldots, t$. We may w.l.o.g. assume that $n_0 = |V(G_1)| = \cdots = |V(G_t)|$ and n_0 divisible by 6, since otherwise we take the least common multiple of 6 and all the graph sizes and apply scalability of the families $\mathcal{F}_i$. Clearly, n_0 can be chosen arbitrarily large as well, such as $n_0 \geq 6(4\ell + t + 4)$. Let $G_0 = M^0_{n_0/6-(\ell+1)}$ (the compensation gadget defined above) and H_0 denote the disjoint union of $\mathcal{K}_\ell$ and G_0. Then $|V(H_0)| = n_0$ and we choose r_0 to be the average degree of H_0;

$$r_0 = \frac{18\ell + 28(n_0/6 - (\ell+1)) + 18}{n_0} = \frac{14n_0/3 - 10(\ell+1)}{n_0}\,.$$

Again by scalability, there exist $G_i^{*a_i} \in \mathcal{F}_i$ (of average degree r_i) such that $|V(G_i^{*a_i})| = a_i n_0$ for $i = 1, \ldots, t$. Similarly, we let $G_0^{*c} = M^{3(c-1)(\ell+1)}_{cn_0/6-c(\ell+1)}$. It is simple calculus to verify that the disjoint union of $\mathcal{K}_\ell$ and G_0^{*c} has cn_0 vertices and the average degree equal to

$$\frac{18\ell + 28(cn_0/6 - c(\ell+1)) + 6 \cdot 3(c-1)(\ell+1) + 18}{cn_0} = \frac{14cn_0/3 - 10c(\ell+1)}{cn_0} = r_0\,.$$

Hence the average degree of the disjoint union of $\mathcal{K}_\ell$ and G_0^{*c} and $G_1^{*a_1}, \ldots G_t^{*a_t}$ indeed is

$$\frac{\sum_{i=1}^{t} a_i n_0 r_i + cn_0 r_0}{\sum_{i=1}^{t} a_i n_0 + cn_0} = r\,. \tag{2}$$

Finally, we let ${G'_0}^{*c} = M^{3(c-1)(\ell+1)+\ell+t}_{cn_0/6-c(\ell+1)}$ and construct the simple 3-connected graph G as the zip product of $\mathcal{K}_\ell$ and ${G'_0}^{*c}$ and $G_1^{*a_1}, \ldots, G_t^{*a_t}$. By Theorem A-2.12, G is crossing-critical with the parameter $\ell + 5 + k_1 + \cdots + k_t = k$, as required. The degrees condition in a) follows from max-universal scalability of $\mathcal{F}_1, \ldots, \mathcal{F}_t$, and b) then follows as well since the size of ${G'_0}^{*c}$ is bounded with respect to c, k. Moreover, by compensation Lemma A-5.3, the average degree of G is equal to r, as in (2). □

Corollary A-5.6 (Lemma 5.3). *Assume D_i-max-universal scalable k_i-crossing-critical families $\mathcal{F}_i$ of average degree r_i, $i = 1, \ldots, t$, as in Lemma A-5.5,*

such that $r_1 < r_2$. Then for every $k \geq k_1 + \cdots + k_t + 5$ and any $r \in (r_1, r_2) \cap \mathbb{Q}$, there exists a $(D_1 \cup \cdots \cup D_t)$-max-universal family of simple, 3-connected, k-crossing-critical graphs of average degree exactly r.

Proof. The proof is a simple exercise in calculus based on Lemma A-5.5. Let $r = \frac{p}{q}$ where p, q are relatively prime integers. Our task is to find infinitely many suitable choices of $a_1, \ldots, a_t$ such that, by (1),

$$\frac{p}{q} = \frac{\sum_{i=1}^{t} a_i r_i + c r_0}{\sum_{i=1}^{t} a_i + c} \tag{3}$$

for some (unknown) rational $r_0 \in (3, 6)$ and suitable (but fixed, see below) c.

To further simplify the task, we choose sufficiently large integer m such that $r_1' = (m r_1 + r_3 + \cdots + r_t)/(m + t - 2) < r$ and set $a_1 = ma$, $a_3 = \cdots = a_t = a$, $a_2 = b$ for yet unknown a, b. Then (3) reads:

$$\frac{p}{q} = \frac{mar_1 + ar_3 + \cdots + ar_t + br_2 + cr_0}{a(m + t - 2) + b + c} = \frac{a(m + t - 2)r_1' + br_2 + cr_0}{a(m + t - 2) + b + c}$$

Let $s = m + t - 2$, and $r_1' = \frac{p_a}{q_a}$, $r_2 = \frac{p_b}{q_b}$, $r_0 = \frac{p_0}{q_0}$. We continue with equivalent processing:

$$\frac{p}{q} = \frac{as\frac{p_a}{q_a} + b\frac{p_b}{q_b} + c\frac{p_0}{q_0}}{as + b + c}$$

$$p(as + b + c)q_a q_b q_0 = asqq_b q_0 p_a + bqq_a q_0 p_b + cqq_a q_b p_0$$

Finally, we get that (3) under our special substitution for $a_1, \ldots, a_t$, is equivalent to the following linear Diophantine equation in a, b:

$$a \cdot sq_b q_0(pq_a - p_a q) + b \cdot q_a q_0(pq_b - p_b q) = cq_a q_b(p_0 q - pq_0)$$

Setting $c = q_0 \cdot GCD\big(sq_b(pq_a - p_a q), q_a(pq_b - p_b q)\big)$, this equation has infinitely many integer solutions, and since $r_1' < r < r_2$, we have that $pq_a - p_a q > 0$ and $pq_b - p_b q < 0$ and so infinitely many of the solutions are among positive integers (regardless of whether the right-hand side is positive, zero or negative). □

Proof (Proof of Theorem A-5.1). The case d) has already been proved in [11], see Proposition A-4.1 a). In all other cases, let $\mathcal{F}_1$ be the family from Proposition A-4.1 c) such that the parameter n satisfies $r_1 = 3 + \frac{1}{4n-7} < r$ (where $r \in A \cap \mathbb{Q}$, $r > 3$, is the desired fixed average degree).

In the case a), let $\mathcal{F}_2$ be a family from Proposition A-4.1 a) with average degree equal to arbitrary (but fixed) $r_2 \in (r, 6) \neq \emptyset$, which can be chosen as scalable by Lemma A-5.4. In the case c), let $\mathcal{F}_2$ be the family from Proposition A-4.1 d) for the parameter ℓ such that $b = 2\ell + 3$; in this case $r_2 = 5 - \frac{8}{b+1} > r$. Finally, we consider the remaining subcases of b). If $D = \{3, 4\}$, then let $\mathcal{F}_2$ be the second family from Proposition A-4.1 a) with average degree $r_2 = 4$. If $D \supsetneq \{3, 4\}$, then let $\mathcal{F}_2$ be the family from Proposition A-4.1 b), made scalable and of fixed average degree $r_2 > 4$ by Lemma A-5.4.

In each one of the choices of $\mathcal{F}_1, \mathcal{F}_2$ above, it holds $r_1 < r < r_2$. Furthermore, if necessary in order to fulfil D-max-universality, we introduce additional scalable families $\mathcal{F}_3, \ldots$ as in the proof of Theorem A-4.3. Theorem A-5.1 then follows directly from Corollary A-5.6. $\square$

A-6 Properties of 2-Crossing-critical Families

In this section, we use the characterization of all 3-connected, 2-crossing-critical graphs from [5].

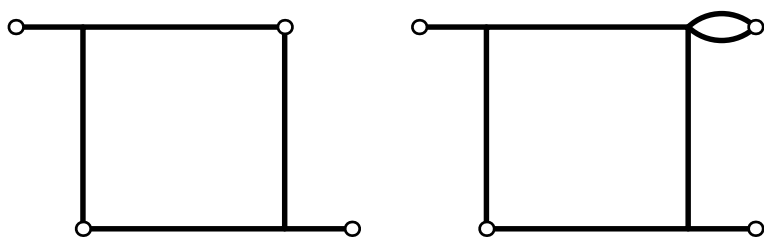


Fig. 8. The two frames.

Definition A-6.1. *The set $\mathcal{S}$ of tiles, obtained as combinations of two frames, illustrated in Figure 8, and* 13 *pictures, shown in Figure 9, in such a way, that a picture is inserted into a frame by identifying the two squares. A given picture may be inserted into a frame either with the given orientation or with a* $180°$ *rotation.*

The set $\mathcal{T}(\mathcal{S})$ consists of all graphs of the form $\circ(\mathcal{T}^{\updownarrow})$, where $\mathcal{T}$ is a sequence $(T_0, {}^{\updownarrow}T_1{}^{\updownarrow}, T_2, \ldots, {}^{\updownarrow}T_{2m-1}{}^{\updownarrow}, T_{2m})$ so that $m \geq 1$ and, for each $i = 0, 1, 2, \ldots, 2m$, $T_i \in \mathcal{S}$.

Operations $\circ$, $\otimes$ and ${}^{\updownarrow}$ on tiles and sequences were defined in Section A-2.

There are examples of some tiles from $\mathcal{S}$ in Figure 10. These tiles are labelled from T_a to T_f and next we will use these labels.

Theorem A-6.2. *([5]) Only finitely many* 3*-connected* 2*-crossing-critical graphs do not contain a subdivision of V_{10} (the graph V_{2n} is obtained from a $2n$-cycle by adding the n main diagonals).*

G is a 3*-connected* 2*-crossing-critical graph containing a subdivision of V_{10} if and only if $G \in \mathcal{T}(\mathcal{S})$.*

Note: A 3-connected, 2-crossing-critical family of graphs contains at most finitely many not-almost-planar graphs because any tile from $\mathcal{T}(\mathcal{S})$ is almost-planar.

We are interested only about infinite family of such graphs, so by this characterization we are interested about graphs which have tile structure and tiles are joining of pictures in Figure 9. Now we can answer our questions about existence of D-max-universal family for 3-connected 2-crossing-critical graphs.

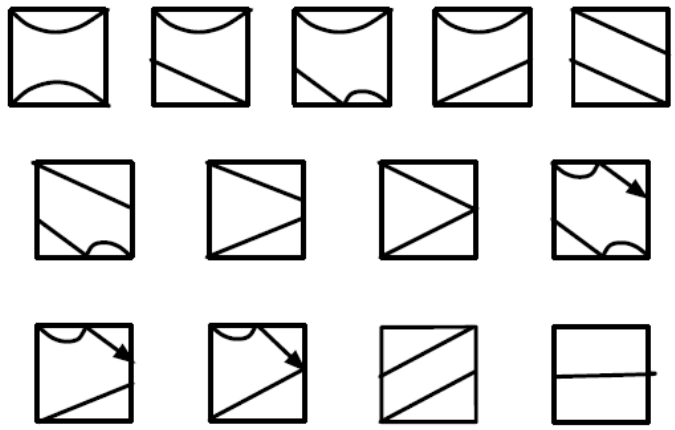


Fig. 9. The thirteen pictures.

Theorem A-6.3 (Theorem 6.1). *A simple,* 3*-connected* 2*-crossing-critical* D*-max-universal family exists if and only if* $\{3\} \subsetneq D \subseteq \{3,4,5,6\}$.

Proof. Let $\mathcal{F}$ be any 3-connected 2-crossing-critical D-max-universal family. By Theorem A-6.2, we may assume $\mathcal{F} \subseteq \mathcal{T}(\mathcal{S})$. There are only nine simple tiles in $\mathcal{S}$ and by join of any two of them we can only construct vertices with degrees 3, 4, 5 and 6, so $D \subseteq \{3,4,5,6\}$. On the other hand, any simple tile from $\mathcal{S}$ has a vertex of degree 3 that is not in its left or right wall, so $\{3\} \subseteq D$, and we get some vertex with degree not equal to 3 after we join any two of them, so $\{3\} \subsetneq D$.

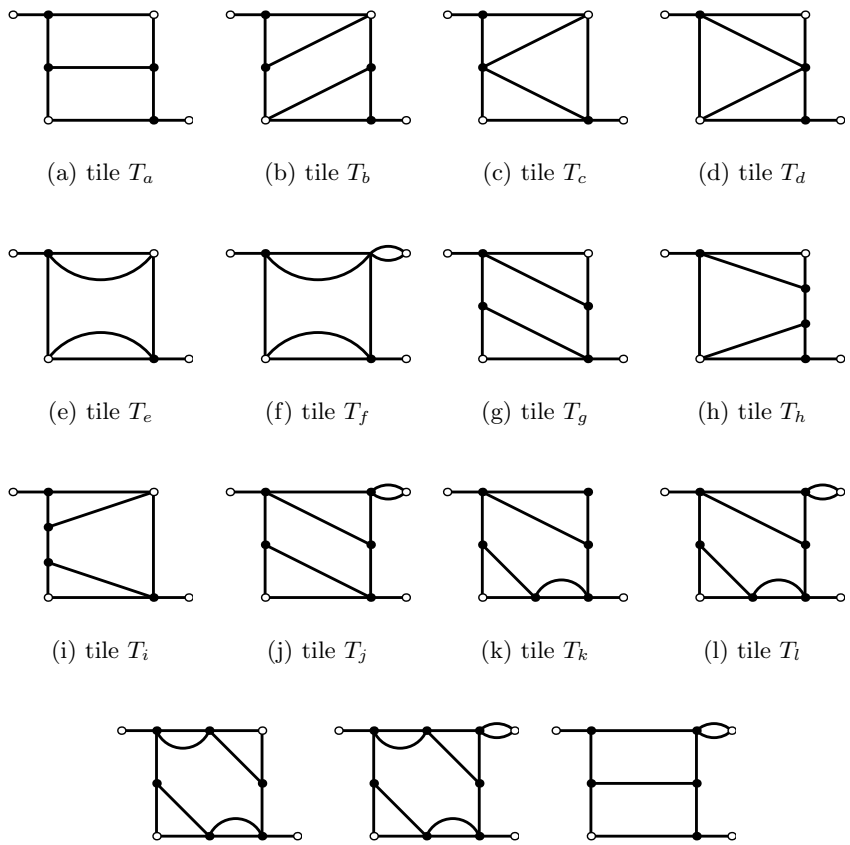
Now we must only construct family $\mathcal{F}$ for any such set D. Consider sequences

$$\begin{aligned}
\mathcal{T}(\{3,4\},m) &= (T_a,{}^{\updownarrow}T_a{}^{\updownarrow},T_a,\ldots,{}^{\updownarrow}T_a{}^{\updownarrow},T_a) \quad \text{(Figure 11)},\\
\mathcal{T}(\{3,5\},m) &= (T_a,{}^{\updownarrow}T_b{}^{\updownarrow},T_a,\ldots,{}^{\updownarrow}T_b{}^{\updownarrow},T_a) \quad \text{(Figure 12)},\\
\mathcal{T}(\{3,6\},m) &= (T_b,{}^{\updownarrow}T_b{}^{\updownarrow},T_b,\ldots,{}^{\updownarrow}T_b{}^{\updownarrow},T_b) \quad \text{(Figure 13)},\\
\mathcal{T}(\{3,4,5\},m) &= (T_a,{}^{\updownarrow}T_d{}^{\updownarrow},T_a,\ldots,{}^{\updownarrow}T_d{}^{\updownarrow},T_a) \quad \text{(Figure 14)},\\
\mathcal{T}(\{3,4,6\},m) &= (T_c,{}^{\updownarrow}T_d{}^{\updownarrow},T_c,\ldots,{}^{\updownarrow}T_d{}^{\updownarrow},T_c) \quad \text{(Figure 15)},\\
\mathcal{T}(\{3,5,6\},2m) &= (T_a,{}^{\updownarrow}T_b{}^{\updownarrow},T_b,{}^{\updownarrow}T_b{}^{\updownarrow},T_a,\ldots,{}^{\updownarrow}T_b{}^{\updownarrow},T_a) \quad \text{combine } \mathcal{T}(\{3,5\},m) \text{ with } \mathcal{T}(\{3,6\},m)\\
\mathcal{T}(\{3,4,5,6\},m) &= (T_c,{}^{\updownarrow}T_b{}^{\updownarrow},T_c,\ldots,{}^{\updownarrow}T_b{}^{\updownarrow},T_c) \quad \text{(Figure 16)},
\end{aligned}$$

where $\mathcal{T}(D,m)$ is a sequence of length $2m+1$ for each set D and positive integer number m.

Let $\{3\} \subsetneq D \subseteq \{3,4,5,6\}$. Then $\circ(\mathcal{T}(D,m)^{\updownarrow})$ is 2-crossing-critical by Theorem A-6.2 and contains vertices only with degrees from D, each at least m times. Hence

$$\mathcal{T}(D) = \{\circ(\mathcal{T}(D,m)^{\updownarrow}); m \in \mathbb{Z}^+ \wedge m \text{ is even when } D = \{3,5,6\}\}$$



(a) tile T_a (b) tile T_b (c) tile T_c (d) tile T_d

(e) tile T_e (f) tile T_f (g) tile T_g (h) tile T_h

(i) tile T_i (j) tile T_j (k) tile T_k (l) tile T_l

(m) tile T_m (n) tile T_n (o) tile T_o

Fig. 10. Examples of tiles from $\mathcal{S}$

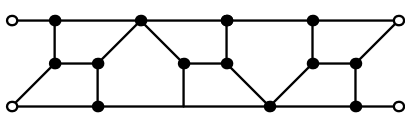


Fig. 11. Tile $\otimes\mathcal{T}(\{3,4\},m)$ for m=1.

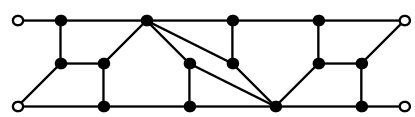


Fig. 12. Tile $\otimes\mathcal{T}(\{3,5\},m)$ for m=1.

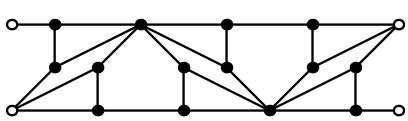


Fig. 13. Tile $\otimes\mathcal{T}(\{3,6\},m)$ for m=1.

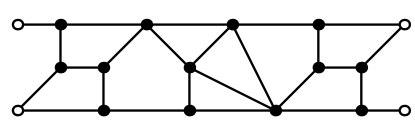


Fig. 14. Tile $\otimes\mathcal{T}(\{3,4,5\},m)$ for m=1.

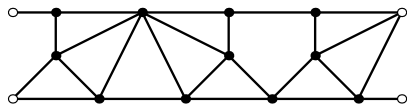


Fig. 15. Tile $\otimes\mathcal{T}(\{3,4,6\},m)$ for m=1.

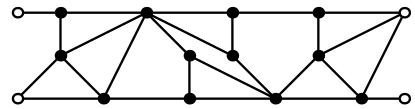


Fig. 16. Tile $\otimes\mathcal{T}(\{3,4,5,6\},m)$ for m=1.

is a 3-connected 2-crossing-critical D-max-universal family. □

Theorem A-6.4 (Theorem 6.1). *A 3-connected, 2-crossing-critical D-max-universal family exists if and only if $D \subseteq \{3,4,5,6\}$, $|D| \geq 2$, and $D \cap \{3,4\} \neq \emptyset$.*

Proof. Let $\mathcal{F}$ be any 3-connected 2-crossing-critical D-max-universal family. By Theorem A-6.2, we may assume $\mathcal{F} \subseteq \mathcal{T}(\mathcal{S})$. These graphs only have degrees 3, 4, 5 and 6, so $D \subseteq \{3,4,5,6\}$. On the other hand, any join of tiles has at least two vertices of different degrees, at least one vertex of them being either 3 or 4, so $|D| \geq 2$ and $D \cap \{3,4\} \neq \emptyset$.

For the converse, we must construct a family $\mathcal{F}$ for any prescribed set D. Using Theorem A-6.3, only sets $D \not\ni 3$ need to be considered. Define:

$$\mathcal{T}(\{4,5\},m) = (T_f,{}^{\updownarrow}T_f{}^{\updownarrow},T_f,\ldots,{}^{\updownarrow}T_f{}^{\updownarrow},T_f) \quad \text{(Figure 17)},$$
$$\mathcal{T}(\{4,6\},m) = (T_e,{}^{\updownarrow}T_e{}^{\updownarrow},T_e,\ldots,{}^{\updownarrow}T_e{}^{\updownarrow},T_e) \quad \text{(Figure 18)},$$
$$\mathcal{T}(\{4,5,6\},m) = (T_e,{}^{\updownarrow}T_f{}^{\updownarrow},T_e,\ldots,{}^{\updownarrow}T_f{}^{\updownarrow},T_e) \quad \text{(Figure 19)},$$

where $\mathcal{T}(D,m)$ is sequence of length $2m+1$ for all set D and positive integer number m.

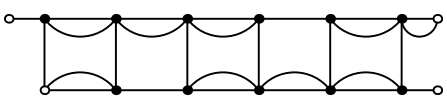


Fig. 17. Tile $\otimes\mathcal{T}(\{4,5\},m)$ for m=1.

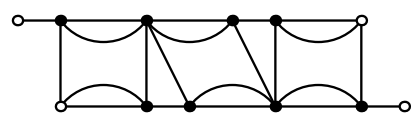


Fig. 18. Tile $\otimes\mathcal{T}(\{4,6\},m)$ for m=1.

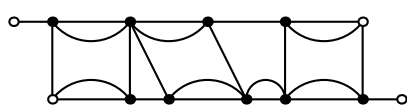


Fig. 19. Tile $\otimes\mathcal{T}(\{4,5,6\},m)$ for m=1.

Let $D \subseteq \{3,4,5,6\}$, $|D| \geq 2$, $D \cap \{3,4\} \neq \emptyset$. Then $\circ(\mathcal{T}(D,m)^{\updownarrow})$ is 2-crossing critical by Theorem A-6.2, contains only vertices of degrees from D, each at least m times. Hence

$$\mathcal{T}(D) = \{\circ(\mathcal{T}(D,m)^{\updownarrow}); m \in \mathbb{Z}^+ \wedge m \text{ is even when } D = \{3,5,6\}\}$$

is a 3-connected 2-crossing-critical D-max-universal family. $\square$

Lemma A-6.5. *Let x, a, b, c_1, c_2, d_1, d_2 are positive integer numbers.*

$$\text{If } x = \frac{a}{b} = \frac{c_1}{d_1} \geq \frac{c_2}{d_2}, \text{ then } x = \frac{a+c_1}{b+d_1} \geq \frac{a+c_2}{b+d_2}.$$
$$\text{Also if } x = \frac{a}{b} = \frac{c_1}{d_1} \leq \frac{c_2}{d_2}, \text{ then } x = \frac{a+c_1}{b+d_1} \leq \frac{a+c_2}{b+d_2}.$$

Proof. If

$$x = \frac{a}{b} = \frac{c_1}{d_1} \geq \frac{c_2}{d_2},$$

then

$$\frac{a+c_1}{b+d_1} = \frac{bx+d_1x}{b+d_1} = \frac{(b+d_1)x}{b+d_1} = x.$$

All numbers are positive so every next modification is equivalent:

$$\begin{aligned} x &\geq \frac{a+c_2}{b+d_2} \\ x(b+d_2) &\geq a+c_2 = xb+c_2 \\ xd_2 &\geq c_2 \\ x &\geq \frac{c_2}{d_2} \end{aligned}$$

Proof for the second inequality is analogous. $\square$

Theorem A-6.6. *A simple, 3-connected, 2-crossing-critical infinite family of graphs with average degree $r \in \mathbb{Q}$ exists if and only if $r \in [3\frac{1}{5}, 4]$.*

Proof. Let T is any tile from $\mathcal{S}$ (from Definition A-6.1). Then $\circ(T^{\updownarrow})$ has v vertices and its sum of degrees is s. The number $\frac{s}{v}$ is the average degree of tile T. If we have any other tile T_1 from $\mathcal{S}$, for which $\circ({T_1}^{\updownarrow})$ has v_1 vertices and its sum of degrees is s_1, then $\circ((T, {}^{\updownarrow}{T_1}^{\updownarrow})^{\updownarrow})$ has average degree $\frac{s+s_1}{v+v_1}$. By Lemma A-6.5, a simple, 3-connected, 2-crossing-critical family with maximum (minimum) average degree exists and contains at most all graphs, which consist of simple tiles with the same maximum (minimum) average degree.

We consider only the simple tiles from $\mathcal{S}$. The tiles T_c and T_d have maximum degree (i.e. 4) and T_a has minimum degree (i.e. $3\frac{1}{5}$). So $\mathcal{T}(\{3,4\})$ (from the proof of Theorem A-6.3) is the family with average degree $3\frac{1}{5}$ and it is minimum and $\mathcal{T}(\{3,4,6\})$ (from the proof of Theorem A-6.3) is the family with average degree 4 and it is maximum.

Now we must only construct simple 3-connected 2-crossing-critical family with average degree $r \in \mathbb{Q}$ for any $r \in (3\frac{1}{5}, 4)$. Let $\frac{p}{q} \in (3\frac{1}{5}, 4)$ (p, q are natural

and coprime) be arbitrary and k is any positive integer number. Then:

$$\begin{aligned} 5p &> 16q & 4q &> p \\ 5p - 16q &> 0 & 16q - 4p &> 0 \\ 5p - 16q - 1 &\geq 0 & 16q - 4p - 1 &\geq 0 \\ 30p - 96q - 6 &\geq 0 & 96q - 24p - 6 &\geq 0 \\ 30p - 96q - 5 &\geq 0 & 96q - 24p - 4 &\geq 0 \\ (30p - 96q - 5)(2k-1) &\geq 0 & (96q - 24p - 4)(2k-1) &\geq 0 \end{aligned}$$

Consider sequence $\mathcal{T}(k) = ({}^{\updownarrow}T_a{}^{\updownarrow}, \ldots, T_a, {}^{\updownarrow}T_a{}^{\updownarrow}, T_b, {}^{\updownarrow}T_b{}^{\updownarrow}, \ldots, T_b, {}^{\updownarrow}T_b{}^{\updownarrow}, T_c, {}^{\updownarrow}T_c{}^{\updownarrow}, \ldots, {}^{\updownarrow}T_c{}^{\updownarrow}, T_c)$, where T_a (together with ${}^{\updownarrow}T_a{}^{\updownarrow}$) is $(96q-24p-4)(2k-1)$-times, T_b (together with ${}^{\updownarrow}T_b{}^{\updownarrow}$) is $8(2k-1)$-times and T_c (together with ${}^{\updownarrow}T_c{}^{\updownarrow}$) is $(30p-96q-5)(2k-1)$-times. Length of this sequence is

$$(96q - 24p - 4)(2k-1) + 8(2k-1) + (30p - 96q - 5)(2k-1) = (6p-1)(2k-1)$$

and it is an odd number.

Graph $\circ({T_a}^{\updownarrow})$ has 5 vertices and its sum of degrees is 16, $\circ({T_b}^{\updownarrow})$ has 5 vertices and its sum of degrees is 18 and $\circ({T_c}^{\updownarrow})$ has 4 vertices and its sum of degrees is 16. So average degree of $\circ(\mathcal{T}(k)^{\updownarrow})$ is

$$\frac{16(96q - 24p - 4)(2k-1) + 18 \cdot 8(2k-1) + 16(30p - 96q - 5)(2k-1)}{5(96q - 24p - 4)(2k-1) + 5 \cdot 8(2k-1) + 4(30p - 96q - 5)(2k-1)} =$$
$$= \frac{-384p - 64 + 480p - 80 + 144}{480q - 20 - 384q - 20 + 40} = \frac{96p}{96q} = \frac{p}{q}.$$

Hence $\{\mathcal{T}(k); k \in \mathbb{Z}^+\}$ is simple 3-connected 2-crossing-critical family with average degree $\frac{p}{q}$. □

Theorem A-6.7. *A 3-connected, 2-crossing-critical infinite family with average degree $r \in \mathbb{Q}$ exists if and only if $r \in [3\frac{1}{5}, 4\frac{2}{3}]$.*

Proof. As in the previous proof, we can prove that 3-connected 2-crossing-critical family with maximum (minimum) average degree exists and contains graphs that consist of tiles with the same maximum (minimum) average degree.

We consider only few tiles from $\mathcal{S}$. From these tiles T_e has maximum degree (i.e. $4\frac{2}{3}$) and T_a has minimum degree (i.e. $3\frac{1}{5}$). So $\mathcal{T}(\{3,4\})$ (from Proof of Theorem A-6.3) is family with average degree $3\frac{1}{5}$ and it is minimum and $\mathcal{T}(\{4,6\})$ (from Proof of Theorem A-6.4) is family with average degree $4\frac{2}{3}$ and it is minimum.

Now we must only construct 3-connected 2-crossing-critical family with average degree $r \in \mathbb{Q}$ for any $r \in (3\frac{1}{5}, 4\frac{2}{3})$. Let $\frac{p}{q} \in (3\frac{1}{5}, 4\frac{2}{3})$ (p, q are natural and

coprime) be arbitrary and k is any positive integer number. Then:

$$\begin{aligned} 5p &> 16q & 14q &> 3p \\ 5p - 16q &> 0 & 14q - 3p &> 0 \\ 5p - 16q - 1 &\geq 0 & 14q - 3p - 1 &\geq 0 \\ 40p - 128q - 8 &\geq 0 & 122q - 24p - 8 &\geq 0 \\ (40p - 128q - 8)(2k-1) &\geq 0 & (122q - 24p - 4)(2k-1) &\geq 0 \end{aligned}$$

Consider sequence $\mathcal{T}(k) = (T_a, {}^{\updownarrow}T_a{}^{\updownarrow}, \ldots, T_a, {}^{\updownarrow}T_a{}^{\updownarrow}, T_c, {}^{\updownarrow}T_c{}^{\updownarrow}, \ldots, {}^{\updownarrow}T_c{}^{\updownarrow}, T_c, {}^{\updownarrow}T_e{}^{\updownarrow}, T_e, \ldots, {}^{\updownarrow}T_e{}^{\updownarrow}, T_e)$, where T_a (together with ${}^{\updownarrow}T_a{}^{\updownarrow}$) is $(112q-24p-4)(2k-1)$-times, T_c (together with ${}^{\updownarrow}T_c{}^{\updownarrow}$) is $11(2k-1)$-times and T_e (together with ${}^{\updownarrow}T_e{}^{\updownarrow}$) is $(40p-128q-8)(2k-1)$-times. Length of this sequence is

$$(112q-24p-4)(2k-1)+11(2k-1)+(40p-128q-8)(2k-1) = (16p-16q-1)(2k-1)$$

and it is odd number.

Graph $\circ(T_a{}^{\updownarrow})$ has 5 vertices and its sum of degrees is 16, $\circ(T_c{}^{\updownarrow})$ has 4 vertices and its sum of degrees is 16 and $\circ(T_e{}^{\updownarrow})$ has 3 vertices and its sum of degrees is 14. So average degree of $\circ(\mathcal{T}(k)^{\updownarrow})$ is

$$\frac{16(112q-24p-4)(2k-1) + 16 \cdot 11(2k-1) + 18(40p-128q-8)(2k-1)}{5(112q-24p-4)(2k-1) + 4 \cdot 11(2k-1) + 3(40p-128q-8)(2k-1)} =$$
$$= \frac{1792q - 384p - 64 + 560p - 1792q - 112 + 176}{560q - 120p - 20 + 120p - 384q - 24 + 44} = \frac{176p}{176q} = \frac{p}{q}.$$

Hence $\{\mathcal{T}(k); k \in \mathbb{Z}^+\}$ is simple 3-connected 2-crossing-critical family with average degree $\frac{p}{q}$. □

Theorem A-6.8. *Let D be such that there exists a D-max-universal 3-connected 2-crossing-critical family. Then let I_D (or I_D^s for simple graphs) is set of all rational numbers, such that there is a D-max-universal 3-connected 2-crossing-critical (simple) family with average degree r if and only if $r \in I_D$ ($r \in I_D^s$). Then I_D^s and I_D are intervals and moreover:*

D	I_D^s	I_D
$\{3,4\}$	$[\frac{16}{5}, \frac{18}{5}]$	$[\frac{16}{5}, \frac{15}{4}]$
$\{3,5\}$	$\{\frac{17}{5}\}$	$[\frac{17}{5}, \frac{11}{3}]$
$\{3,6\}$	$\{\frac{18}{5}\}$	$\{\frac{18}{5}\}$
$\{4,5\}$	$\emptyset$	$\{\frac{9}{2}\}$
$\{4,6\}$	$\emptyset$	$\{\frac{14}{3}\}$
$\{3,4,5\}$	$(\frac{16}{5}, 4]$	$(\frac{16}{5}, 4]$
$\{3,4,6\}$	$(\frac{16}{5}, 4]$	$(\frac{16}{5}, \frac{14}{3})$
$\{3,5,6\}$	$(\frac{17}{5}, \frac{18}{5})$	$(\frac{17}{5}, \frac{11}{3})$
$\{4,5,6\}$	$\emptyset$	$(\frac{9}{2}, \frac{14}{3})$
$\{3,4,5,6\}$	$(\frac{16}{5}, 4]$	$(\frac{16}{5}, \frac{14}{3})$