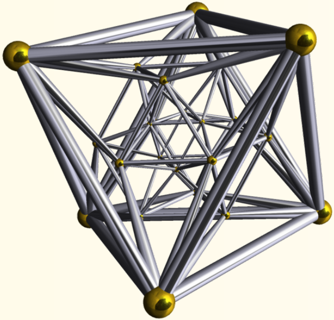
A Short Proof of Euler–Poincaré Formula



Petr Hliněný

Faculty of Informatics, Masaryk University

Brno, Czech Republic

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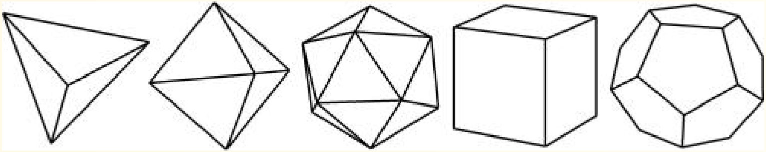

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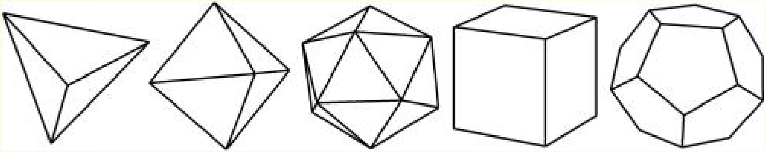
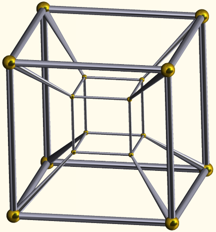


- The first landmark in the theory of polytopes.
- Known already to Descartes. First full proof by Legendre in 1794.
- See also David Eppstein: *Twenty Proofs of Euler's Formula*.
  The Geometry Junkyard `http://www.ics.uci.edu/~eppstein/junkyard/euler`.

2 Schläfli: Higher Dimensions

Theorem. Let P be a convex polytope in $\mathbb{R}^d$, and denote by $\boldsymbol{f^c}$, $c \in \{0, 1, \ldots, d\}$, the numbers of *faces of P of dimension c*. Then

$$\boldsymbol{f^0 - f^1 + f^2 - \cdots + (-1)^{d-1} f^{d-1} + (-1)^d f^d = 1}$$



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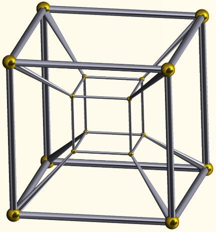

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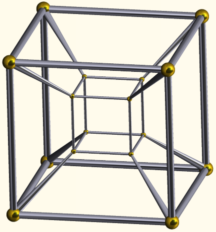

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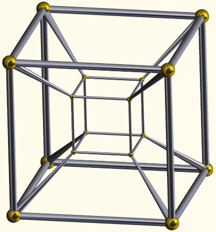

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- Rediscoveries in late 19th century, all proofs assuming *shellability*.
- Shellability established only in 1971 by Bruggesser and Mani.

3 New Proof: setup

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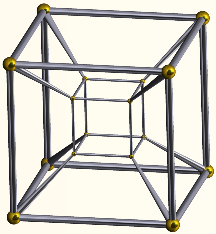
or, $f^0 - f^1 + \cdots + (-1)^{d-1} f^{d-1} = \mathbf{1 + (-1)^{d-1}}$



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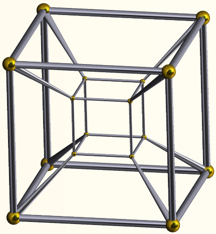

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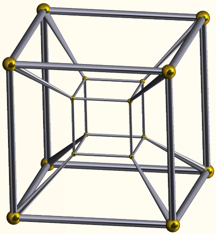

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- $\boldsymbol{R :=}$ the *Schlegl diagram* of P (a complex in $\mathbb{R}^{\boldsymbol{k}}$), having the facets $R_1, \ldots, R_t$ where $t := f^k - 1$, and R_0 being the "outer" facet of the diagram.

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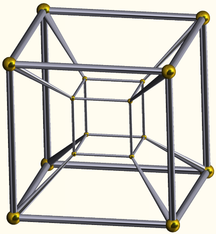

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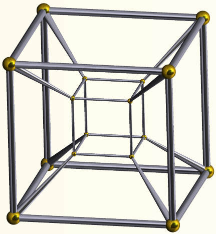

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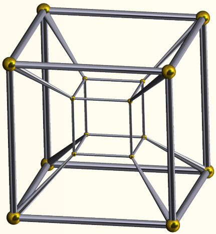

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  Let $\boldsymbol{f_i^c}$ be the number of faces of R_i of dimension c.
- Choose a general direction (vector) α in $\mathbb{R}^k$, i.e., one not parallel to any proper face in the complex R.
- Assign two *flags* to each face of dim. $< k$, one as α and one as $-\alpha$.
  Formally, the flags are $\varepsilon\alpha$ and $-\varepsilon\alpha$ for small $\varepsilon > 0$, starting both in a point in the relative interior of this face.

New Proof: counting flags

Recall the setup:

- $f^0 - f^1 + \cdots + (-1)^{d-1} f^{d-1} = 1 + (-1)^{d-1}$, for $d \leq k$,
- P in dimension $d := k + 1$, $R :=$ Schlegl diagram of P,
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- *Globally* – all flag values together: (cf. Schläfli!)

$$2\left(f^0 - f^1 + \cdots + (-1)^{k-1} f^{k-1}\right) = 2 \sum_{c=0}^{k-1} (-1)^c f^c$$

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- *Locally* – every flag "belongs" to precis. one of the facets in R...
  (flags pointing out of R belong to outer R_0)

New Proof: counting locally

Recall the **global** sum

$$\text{“}\textstyle\sum \text{ all flags”} = 2\sum_{c=0}^{k-1}(-1)^c f^c \quad = 2 \times \text{“Schläfli”}\,.$$

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slightly different. . .

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Towards the conclusion. . .

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New Proof: wrapping up

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Putting together

$$2\sum_{c=0}^{k-1}(-1)^c f^c = \sum_{a=1}^{t} 2(-1)^{k-1} + 2$$

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$$\sum_{c=0}^{k}(-1)^c f^c = -1\cdot(-1)^{k-1} + 1 = 1 + (-1)^k$$

□

4 Conclusions

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Thank you for your attention.

and

Long live the ACO!