Planar Emulators Conjecture Is Nearly True for Cubic Graphs

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Abstract

We prove that a cubic nonprojective graph cannot have a finite planar emulator, unless it belongs to one of two very special cases (in which the answer is open). This shows that Fellows' planar emulator conjecture, disproved for general graphs by Rieck and Yamashita in 2008, is nearly true on cubic graphs, and might very well be true there definitely.

Keywords: planar emulator; projective planar graph; graph minor

1. Introduction

A graph G has a finite *planar emulator* H if H is a planar graph and there is a graph homomorphism $\varphi : V(H) \to V(G)$ where φ is locally surjective, i.e. for every vertex $v \in V(H)$, the neighbours of v in H are mapped surjectively onto the neighbours of $\varphi(v)$ in G. We also say that such a G is planar-emulable. If we insist on φ being locally bijective, we get H a *planar cover*.

The concept of planar emulators was proposed in 1985 by M. Fellows [6], and it tightly relates (although of independent origin) to the better known *planar cover conjecture* of Negami [11]. Fellows also raised the main question: What is the class of graphs with finite planar emulators? Soon later he conjectured that the class of planar-emulable graphs coincides with the class of graphs with finite planar covers (conjectured to be the class of projective graphs by Negami [11]—still open nowadays). This was later restated as follows:

Conjecture 1 (M. Fellows, falsified in 2008). *A connected graph has a finite planar emulator if and only if it embeds in the projective plane.*

For two decades, the research focus was nearly exclusively on Negami's conjecture and no substantial new results on planar emulators had been presented

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until 2008, when emulators for two nonprojective graphs were given by Rieck and Yamashita [13], effectively disproving Conjecture 1.

Planar-emulable nonprojective graphs. Following Rieck and Yamashita, Chimani et al [2] constructed finite planar emulators of all the minor minimal obstructions for the projective plane except those which have been shown non-planar-emulable already by Fellows (the $K_{3,5}$ and "two disjoint k-graphs" cases, Section 2), and except $K_{4,4} - e$. The graph $K_{4,4} - e$ is thus the only forbidden minor for the projective plane where the existence of a finite planar emulator remains open. Even though we do not have a definite replacement for falsified Conjecture 1 yet, the results obtained so far [5, 2] suggest that, vaguely speaking, up to some trivial operations there is only a finite family of nonprojective planar-emulable graphs. A result like that would nicely correspond with the current state-of-art [10] of Negami's conjecture.

The aforementioned trivial operation(s) is the following one: A *planar expansion* of a graph G is a graph which results from G by repeatedly (i) adding a planar graph P sharing one vertex with G, or by replacing (ii) an edge v_1v_2 or (iii) a cubic vertex with neighbours v_1, v_2, v_3 by a connected planar graph P, such that P shares with G precisely the vertices v_1, v_2 (v_1, v_2, v_3, respectively) which lie on the outer face of P.

While characterization of planar-emulable graphs has proven itself to be difficult in general, significant progress can be made in a special case. Negami's conjecture has been confirmed in the case of cubic graphs in [12], and the same readily follows from [10]. Here we prove:

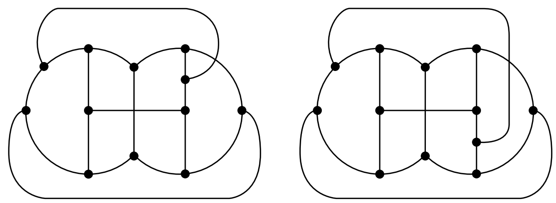
Theorem 2. *If a cubic nonprojective graph H has a finite planar emulator, then H is a planar expansion of one of two minimal cubic nonprojective graphs shown in Figure 1.*

A computerized search for possible counterexamples to Conjecture 1, carried out so far [5], shows that a nonprojective planar-emulable graph G cannot be cubic, unless G contains a minor isomorphic to $\mathcal{E}_2$, $K_{4,5} - 4K_2$, or a member of the so called "$K_7 - C_4$ family" (see [2] for the terminology). Our new "hand-written" approach, Theorem 2, dismisses the former two possibilities completely and strongly restricts the latter one.

2. Preliminaries

We consider standard terms of graph theory, and deal only with simple finite graphs. A graph of all degrees equal 3 is called *cubic*. The following definition turns out very useful: A graph G is said to *contain two disjoint k-graphs* if there exist two vertex-disjoint subgraphs $J_1, J_2 \subseteq G$ such that, for $i = 1, 2$, the graph J_i is isomorphic to a subdivision of K_4 or $K_{2,3}$, the subgraph $G - V(J_i)$ is connected and adjacent to J_i, and contracting in G all the vertices of $V(G) \setminus V(J_i)$ into one results in a nonplanar graph.

We use some folklore known facts about planar-emulable graphs (also [2]):



G_1 G_2

Figure 1: Two (out of six in total) cubic irreducible obstructions for the projective plane [7]. Although G_1 and G_2 result from splitting nonprojective graphs for which we have finite planar emulators [2] (namely "relatives" of $K_7 - C_4$), it is still open whether G_1 and G_2 themselves are planar-emulable.

Proposition 3 (Fellows, unpublished). *Let G be a connected graph.*

a) *The class of planar-emulable graphs is closed under taking minors.*
b) *If G is projective and connected, then G has a finite planar emulator in form of its finite planar cover.*
c) *If G contains two disjoint k-graphs or a $K_{3,5}$-minor, then G is not planar-emulable.*
d) *G is planar-emulable if, and only if, so is any planar expansion of G.*

Using Glover and Huneke [7], one obtains the following starting point for our proof of Theorem 2.

Proposition 4. *Let G_1, G_2 be the graphs from Figure 1. If a cubic nonprojective graph H has a finite planar emulator, then H contains a subgraph $G' \subseteq H$ being a subdivision of $G \in \{G_1, G_2\}$.*

Proof. [7] characterized the cubic graphs with projective embedding by giving a set $\mathcal{I}$ of six cubic graphs such that: if H is a cubic graph that does not embed in the projective plane, then H contains a graph $G \in \mathcal{I}$ as a topological minor.

Let us point out that four out of the six graphs in $\mathcal{I}$ contain two disjoint k-graphs, and so only the remaining two— $G_1 \in \mathcal{I}$ and $G_2 \in \mathcal{I}$ of Figure 1, can potentially be planar-emulable. Hence the cubic graph H contains one of G_1, G_2 as a topological minor. In other words, there is a subgraph $G' \subseteq H$ being a subdivision of cubic $G \in \{G_1, G_2\}$. □

3. Planar and non-planar expansions

The purpose of this section is to classify and analyze the difference $H - G'$ where G', H are from Proposition 4. In order to do so, we need to introduce some basic technical concepts.

We call a *bridge of G' in H* any connected component B of $H - V(G')$ together will all the incident edges. In a degenerate case, B might consist just of one edge from $E(H) \setminus E(G')$ with both ends in G'. We would like, for

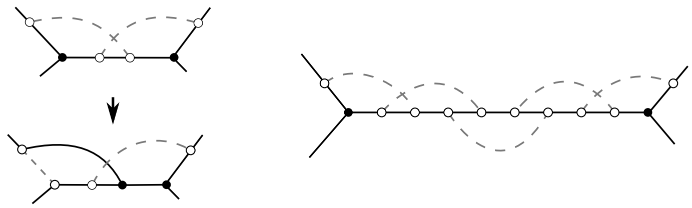


Figure 2: Illustration for Lemma 5. The trivial bridge on the left takes over the role of a branch vertex of G in the subdivision, resulting in existence of a nontrivial bridge. The other picture shows when the transitive closure of declared attachment becomes important.

simplicity, to speak about positions of bridges with respect to the underlying cubic graph G: Such a bridge B connects to vertices u of G' which subdivide edges f of G—this is due to the cubic degree bound, and we (with neglectable abuse of terminology) say that B *attaches to* this edge f in G itself.

A bridge B is *nontrivial* if B attaches to some (at least) two nonadjacent edges of G, and B is *trivial* otherwise. For a trivial bridge B; either B attaches to only one edge f in G and we say *exclusively to* f, or all the edges to which B attaches in G have a vertex w in common (since G contains no triangles), and we say that B *attaches to* this w.

We divide the rest of the analysis into two main cases; that either some bridge of G' in H is nontrivial or all such bridges are trivial. We moreover assume that $G' \subseteq H$ being a subdivision of G is chosen in Proposition 4 such that it *has a nontrivial bridge* if possible.

In the "all-trivial" case one more technical condition has to be observed: Suppose B_1, B_2 are bridges such that B_1 attaches to w and B_2 attaches to an edge f incident to w in G (perhaps B_2 exclusively to f). On the path P_f which replaces (subdivides) f in G', suppose that B_2 connects (i.e., is adjacent) to some vertex which is closer to w on P_f than some other vertex which B_1 connects to. Then we *declare that* B_2 *attaches to* w, too, and we make a transitive closure of this declaration on whole G. This is well defined with respect to our aforementioned assumption (that no $G' \subseteq H$ has a nontrivial bridge) because of the following:

Lemma 5. *Let $G' \subseteq H$ be a subdivision of G where G, H are cubic graphs. Suppose that all bridges of G' in H are trivial, and that a bridge B_0 is declared to attach to both w_1 and w_2, where $w_1 w_2 \in E(G)$. Then there is $G'' \subseteq H$ which is isomorphic to a subdivision of G, too, and a nontrivial bridge of G'' in H exists.*

Proof. We refer to an informal sketch in Figure 2. Let P_f be the path representing $f = w_1 w_2$ in $G' \subseteq H$. In the described situation, we call B_0 a *conflicting* bridge of G' and assume, by minimality, that $H - B_0$ has no conflicting bridge of G'. By this assumption and the definition of declared attachment there exist vertices $u_1, u_2 \in V(P_f)$ such that the following holds for $i = 1, 2$:

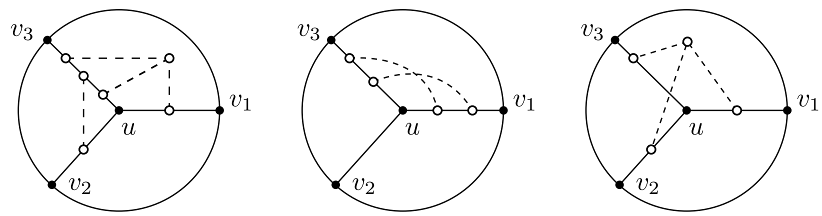


Figure 3: Illustration for Lemma 6; three collections of trivial bridges that attach to a cubic vertex u (with neighbours v_1, v_2, v_3 in G). The first collection gives a planar expansion, while the other two informally outline the "minimal" non-planar-expansion cases.

a) either $u_i = w_i$ and B_0 attaches to w_i in the primary sense, i.e., that B_0 attaches to at least two of the edges incident to w_i, or
b) there is another bridge B_i connecting to u_i such that B_i attaches, or is declared to attach, to w_i in G, and B_0 connects the two components of $P_f - u_i$ together.

Notice in b) that $u_1 \neq u_2$ since H is cubic, and that $B_1 \neq B_2$ and u_1 is closer to w_1 on P_f than u_2 since $H - B_0$ has no conflicting bridge.

Let $f_i' = w_i w_i'$ and $f_i'' = w_i w_i''$ be the edges of G distinct from f and $P_{f_i'}, P_{f_i''}$ be the corresponding paths in G'. Assume b) has happened for $i = 1$, and let A_1 be the union of all the bridges except B_0 which attach to f and which, moreover, connect to a vertex on P_f between w_1 and u_1 (these include B_1). Each bridge in A_1 is declared to attach to w_1 by the definition. Since $H - B_0$ has no conflicting bridge, no vertex of A_1 is adjacent to a vertex of P_f closer to w_2 than u_2. We claim that $X_1 = A_1 \cup P_f \cup P_{f_1'} \cup P_{f_1''}$ contains internally disjoint paths from u_1 to each of w_1' and w_1''. Indeed, a cutvertex between u_1 and w_1', w_1'' in X_1 would certify that B_1 is not declared to attach to w_1, a contradiction.

The same about internally disjoint paths can be claimed for u_2 and w_2', w_2''. We can now define $G'' \subseteq H$ being isomorphic to a subdivision of G, such that G'' coincides with G' everywhere on $V(G)$ except that the branch vertices $w_1, w_2 \in V(G)$ are replaced with aforementioned u_1, u_2. Now, B_0 makes a nontrivial bridge of G''. $\square$

Lemma 6. *Let $G' \subseteq H$ be a subdivision of G where G is a 2-connected cubic nonplanar graph which does not contain two disjoint k-graphs. Suppose that all bridges of G' in H are trivial, and none of them is conflicting (cf. Lemma 5). Then H does not contain two disjoint k-graphs if, and only if, H is a planar expansion of G.*

Proof. Assume H is a planar expansion of G, and yet H contains two disjoint k-graphs. Let $J_1, J_2 \subseteq H$ be these two "k-graphs" by the definition, and denote by L_i, $i = 1, 2$, the nonplanar graphs obtained by contracting all vertices of $V(H) \setminus V(J_i)$ into one. Let L_i' be the corresponding minors of G'. Then each L_i' is nonplanar as well since L_i is its planar expansion. Consequently, J_i or other

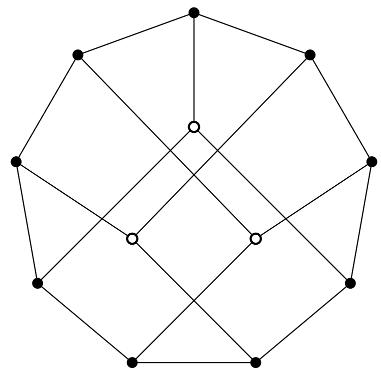


Figure 4: Another picture of the graph G_1 from Proposition 4.

subdivision of K_4 or $K_{2,3}$ is contained in G', and hence G' and G contain two disjoint k-graphs, a contradiction.

In the converse direction, assume that H is not a planar expansion of G. Let B_v be the union of all trivial bridges of G' in H that attach or are declared to attach to a vertex $v \in V(G)$. Let B_f be the union of all trivial bridges of G' in H that attach exclusively to an edge $f \in E(G)$ and are not declared to attach to either of its ends. Since H is not a planar expansion of G and no bridge is conflicting, for at least one $x \in V(G) \cup E(G)$ the subgraph $H_x = G' \cup B_x$ is not a planar expansion of G, too.

We start with the more interesting case $x = u \in V(G)$. See an illustration in Figure 3. Let $G'_u \subseteq G'$ denote the corresponding subdivision of $G - u$. Let C be a 3-edge-cut in H_u which separates G'_u on one side and $B'_u \supset V(B_u) \cup \{u\}$ on the other side. Then, by the definition, our graph H_u is not a planar expansion of G' iff B'_u is not planar with all the three connections to C on the outer face. The latter is characterized by containment of a $K_{2,3}$-subdivision $J_1 \subseteq B'_u$ with the size-three part incident to C. Moreover, since G is cubic nonplanar, there exists a $K_{3,3}$-subdivision $J_0 \subseteq G'$, and hence a $K_{2,3}$-subdivision $J_2 \subseteq G'_u$. Then J_1, J_2 certify that H_u (and so H as well) contains two disjoint k-graphs.

For the case of $x = f \in E(G)$ we proceed in a similar way with the change that C is a 2-edge cut and B'_u is not planar with both connections to C on the outer face. This even stronger condition again implies the existence of $K_{2,3}$-subdivisions $J_1 \subseteq B'_u$ and $J_2 \subseteq G'_u$, and we finish in the same way. □

4. Analysis of nontrivial bridges

In view of Lemma 6, it remains to investigate what happens if G' from Proposition 4 has a nontrivial bridge in H. Originally we have done this case check on a computer, but here we present a relatively easy handwritten approach. We think it is worth the investigation since it discovers some beautiful symmetries of the graphs G_1, G_2.

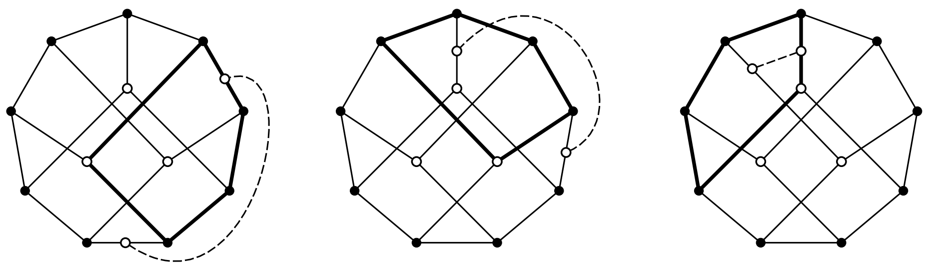


Figure 5: Illustrating examples for the proof of Lemma 7: in each of the cases we see in bold one of the "magic" 5-cycles of G_1 forming a subdivision of $K_{2,3}$ with attached dashed edge f. The types of these cases are I, II, I from left to right.

Lemma 7. *Let $G' \subseteq H$ be a subdivision of G_1 (from Proposition 4) where H is a cubic graph. If there exists a nontrivial bridge of G' in H, then H contains two disjoint k-graphs.*

Proof. Without loss of generality we may assume that H results from G' by adding a single nontrivial bridge, which is an edge f attached to edges $e_1, e_2 \in E(G_1)$. To easily analyse the possible cases by hand, we use a "nice" picture of the graph G_1 in Figure 4. Note that the picture consists of 9 *rim edges* (incident to the outer face) and 9 *spoke edges*. Another useful observation is that each of the 9 incident pairs of spoke edges defines, together with three rim edges, a *magic* 5-cycle in G_1, such that its complement is a 7-vertex subdivision of $K_{2,3}$. Our aim is to show that each possible edge f attaches to two edges incident with some of the magic 5-cycles in a way giving us another 7-vertex subdivision of $K_{2,3}$. Those two together then show that H contains two disjoint k-graphs.

In the rest of this proof we analyze all possible attachments of the nontrivial bridge f to edges $e_1, e_2 \in E(G_1)$. First, with respect to any one of the magic 5-cycles C, there exist essentially two possibilities (*types*) which always lead to a desired subdivision of $K_{2,3}$:

I. $e_1 \in E(C)$ and $e_2 \notin E(C)$ is incident to one of the three remaining vertices of C, or

II. $e_1, e_2 \notin E(C)$ are both incident to two nonadjacent vertices of C.

As for the position of f within G_1, there are altogether at most 10 possibilities, up to symmetry (see Figures 4 and 5). We group them as follows.

a) Both e_1, e_2 are rim edges, which includes three cases having 1, 2, and 3 other rim edges between e_1, e_2 on the outer cycle. One can easily check from the picture that the first two are of type I and the third is of type II.

b) e_1 is a spoke edge, unique up to symmetry, and e_2 is a rim edge. This includes four cases based on the distance 1, 2, 3, and 4 of e_2 from the rim vertex of e_1. The first two cases are (not exclusively) of type II while the other are both of type I.

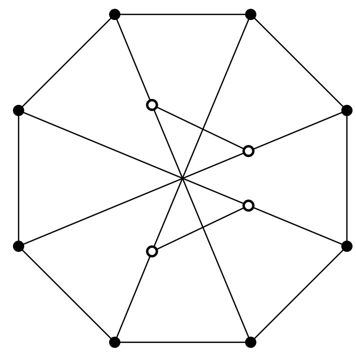


Figure 6: Another picture of the graph G_2 from Proposition 4.

c) Both e_1, e_2 are spoke edges, which makes three cases based on the distance 1, 2, and 4 of the rim vertices of e_1, e_2 on the rim cycle. While the first two cases are of type I, the third is of type II.

In all the nonsymmetric cases we have thus found that H contains two disjoint k-graphs. □

Lemma 8. *Let $G' \subseteq H$ be a subdivision of G_2 (Proposition 4) where H is a cubic graph. If there exists a nontrivial bridge of G' in H, then H contains two disjoint k-graphs or a $K_{3,5}$-minor.*

Proof. We start as in Lemma 7, and give another "nice" picture of the graph G_2 in Figure 6. Note that the picture consists of 8 *rim edges* (incident to the outer face), 8 *spoke edges* and 2 *bar edges* in the middle. Again, there are 8 *magic* 5-cycles, each formed by a pair of the spoke edges with a common bar edge and corresponding two of the rim edges. The complement of each magic cycle contains a 7-vertex subdivision of $K_{2,3}$, too. We hence, whenever possible, aim to use the same tools as in the proof of Lemma 7.

For the position of the bridge f within G_2 there exist altogether at most 12 possibilities, up to symmetry. We group them as follows.

a) Both e_1, e_2 are rim edges, which includes three cases having 1, 2, and 3 other rim edges between e_1, e_2 on the outer cycle.
b) e_1 is a spoke edge, unique up to symmetry, and e_2 is a rim edge. This includes three cases based on the distance 1, 2, and 3 of e_2 from the rim vertex of e_1.
c) Both e_1, e_2 are spoke edges, making three distinct cases; where in one of them e_1, e_2 have a common incident bar edge.
d) e_1 is a bar edge; there exist three such cases having e_2 as the other bar edge, e_2 as a rim edge, and e_2 as a spoke edge not incident to e_1.

Out of these 12 cases, six (two from each) of a), b), d) and all three of c) are solved exactly as in Lemma 7: these cases are each of type I or type II with respect to some magic 5-cycle in G_2, finally giving two disjoint k-graphs in H.

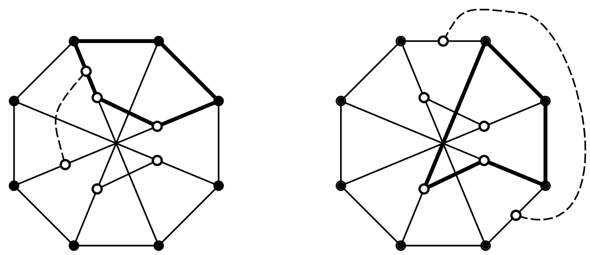


Figure 7: Illustrating examples for the proof of Lemma 8: we see in bold one of the “magic” 5-cycles of G_2 forming a subdivision of $K_{2,3}$ (left of type I, right of type II) with attached dashed edge f.

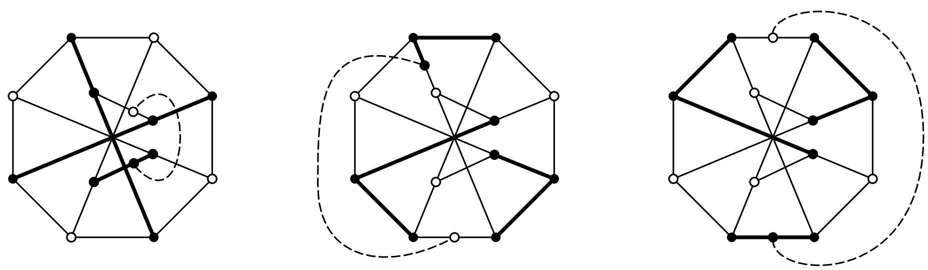


Figure 8: More examples for the proof of Lemma 8: in each of the cases there is an independent set of 5 hollow vertices, such that a $K_{3,5}$-minor (with the large part on these hollow vertices) results by contracting the three paths in bold.

Instead of boring repetition of the details we just give a short illustrating picture in Figure 7.

In the three remaining cases we apply the following easy observation: in any 14-vertex triangle-free cubic graph, if there is an independent set of size 5 such that its complement consists of three induced paths of length 2, no one of these paths being part of a 4-cycle, then contracting the three paths makes a minor isomorphic to $K_{3,5}$. An application of this claim to our three cases in question is depicted in Figure 8. $\square$

Proof (of Theorem 2). Firstly, H cannot contain two disjoint k-graphs or a $K_{3,5}$-minor by Proposition 3 c). Hence, if $G' \subseteq H$ being a subdivision of $G \in \{G_1, G_2\}$ (cf. Proposition 4) could be chosen such that a nontrivial bridge of G' exists in H, then there would be a contradiction. Consequently, all bridges of G' in H are trivial and, by Lemma 5, they are not conflicting. Then H is a planar expansion of G by Lemma 6. $\square$

5. Conclusions

We identified two graphs G_1, G_2 (Figure 1), for which existence of finite planar emulator now becomes extremely interesting. We would like to point out that similarity of these two graphs, seen also in the proofs of Lemmas 7 and 8, suggests that if one has a finite planar emulator, so does the other one.

We suggest that the nicely symmetric structure of the "magic" cycles in each of G_1, G_2, shown in Section 4 could be a strong starting point in a possible proof that G_1, G_2 do not have finite planar emulators. We continue an intensive research in this direction. Overall, we believe that providing an answer for any of these two graphs G_1, G_2 would bring a better insight to the problem of planar emulations not only for the cubic case, but also in general.

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