Twin-width of Planar Graphs is at most ~~11~~ 9

Petr Hliněný

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Brno, Czech Republic

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- **Contraction sequence** – a sequence of simple contractions of vertex pairs (arbitrary pairs, unlike in graph minors!);
  - a contraction of a pair makes an edge red if it existed to one of the contracted vertices but not to the other, and
  - red edge stays red till the end.

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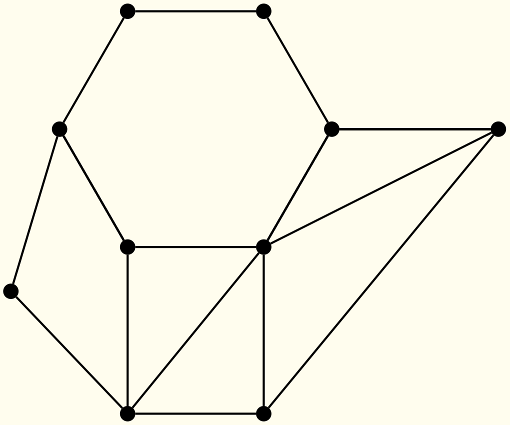

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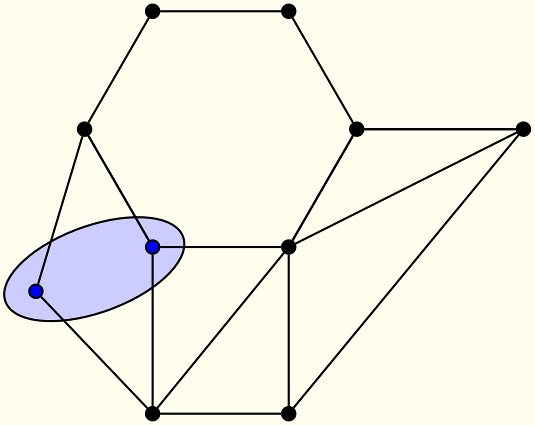

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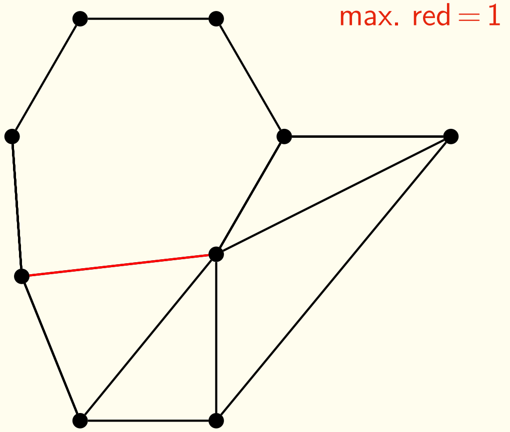

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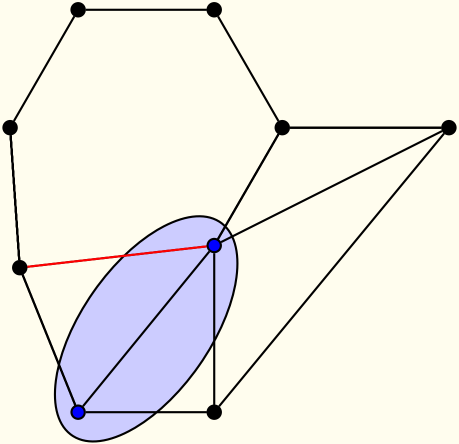

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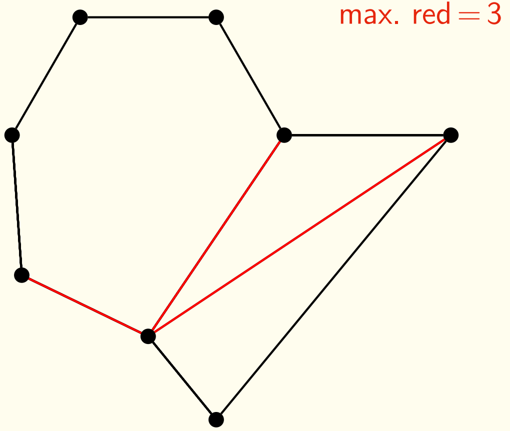

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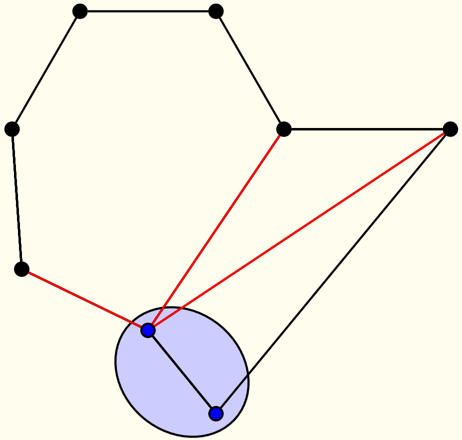

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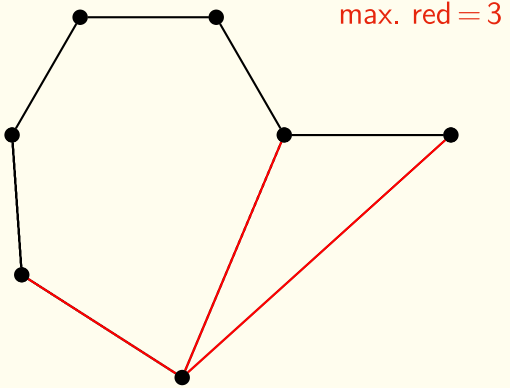

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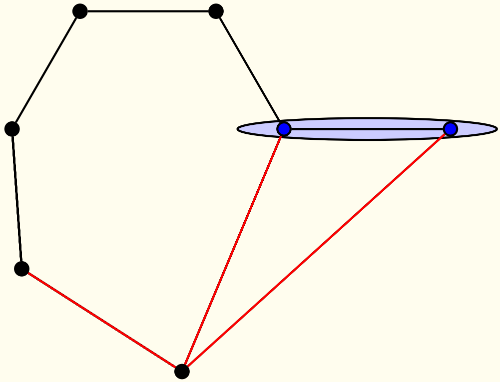

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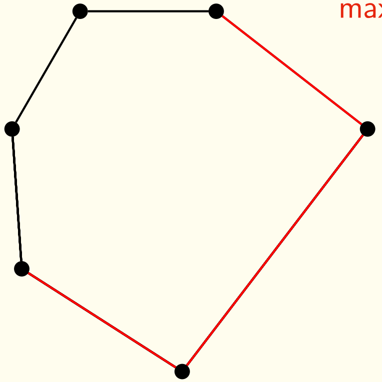

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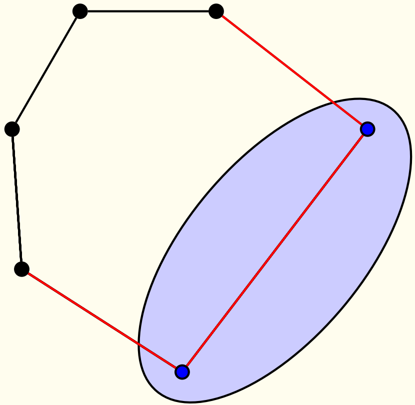

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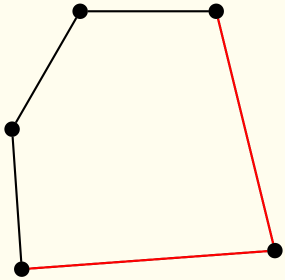

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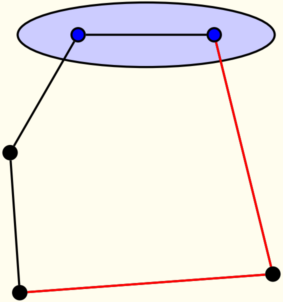

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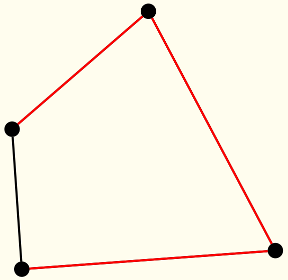

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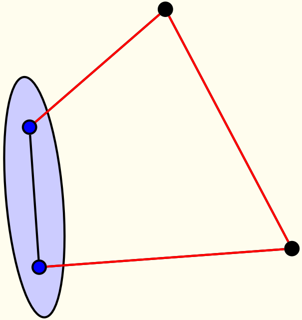

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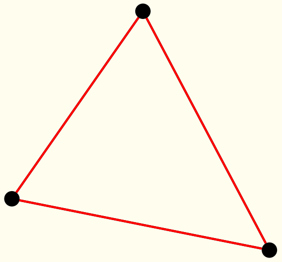

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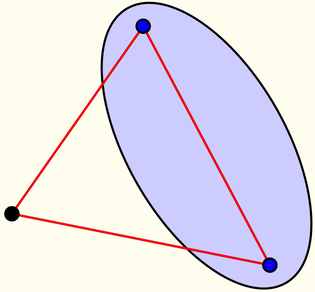

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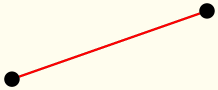

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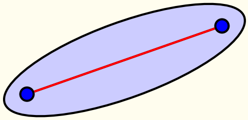

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twin-width ≤ 3

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  this new concept seems to be crucial in the ongoing quest to characterise hereditary classes with tractable FO model checking (cf. the subsequent talks. . .).

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- cubic graphs (!!!).

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≥ 5 quite easily, but no better lower bound published so far...

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- Choose a root on the outer f., and a BFS tree of G^+ from this root. *Note that all edges are only between same and successive BFS layers.*
- Formulate a suitable (recursive) claim about partial contractions inside a bounded region of the plane triangulation. *Prove by induction.*

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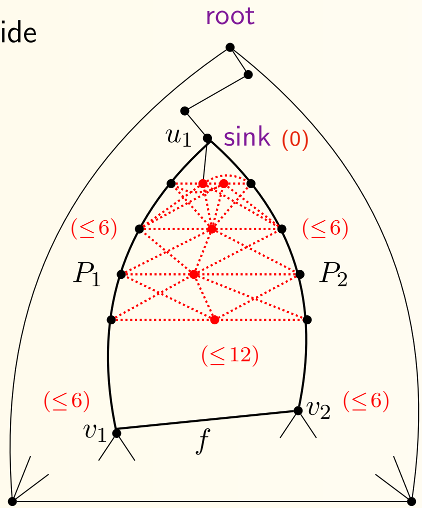

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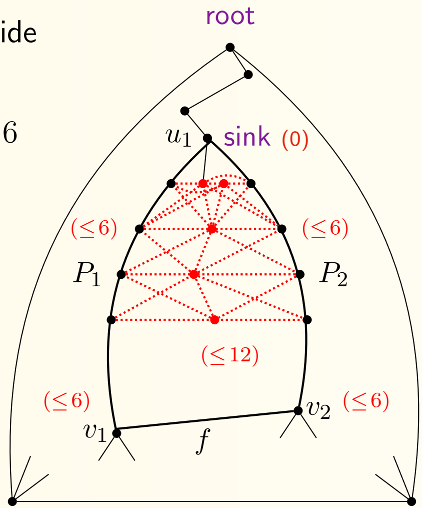

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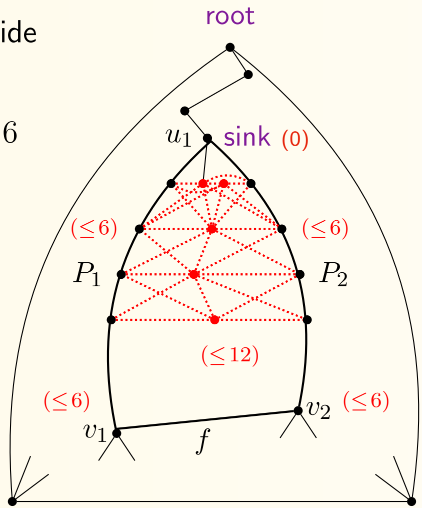

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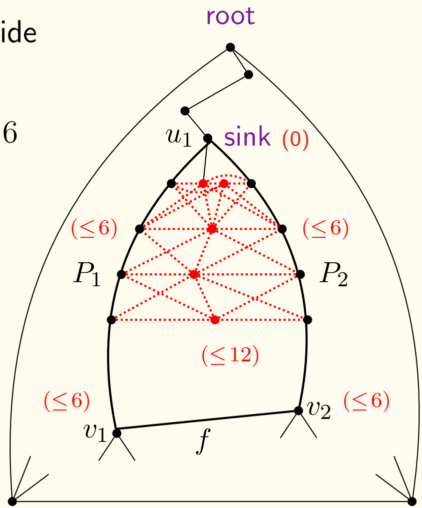

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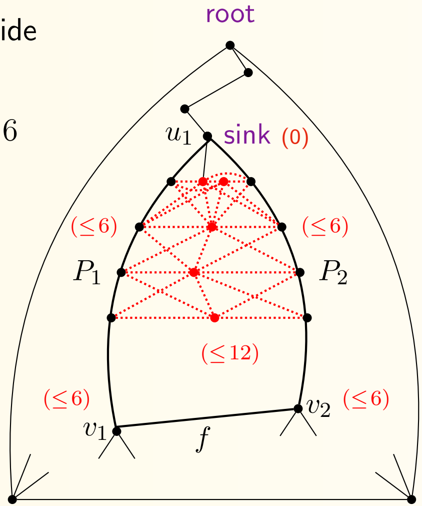
- after the contractions, each BFS layer inside has only 1 vertex, except ≤ 2 vert. next to the sink.



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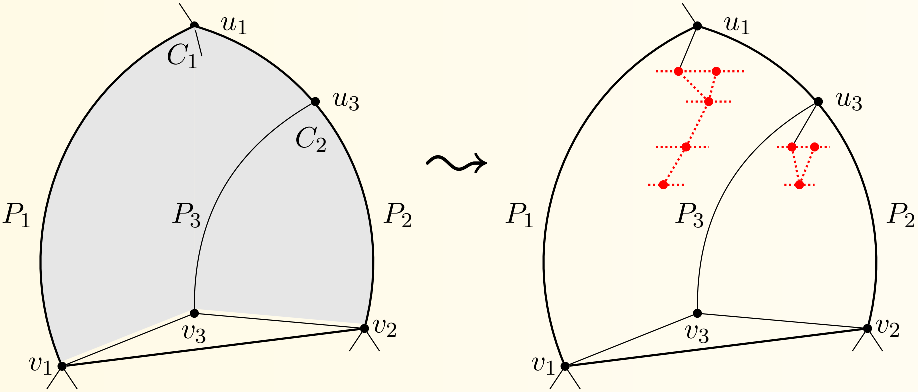
- Apply the Lemma inductively to each of the two subregions:



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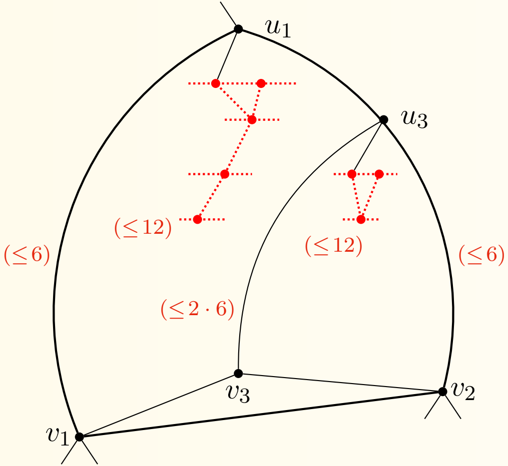

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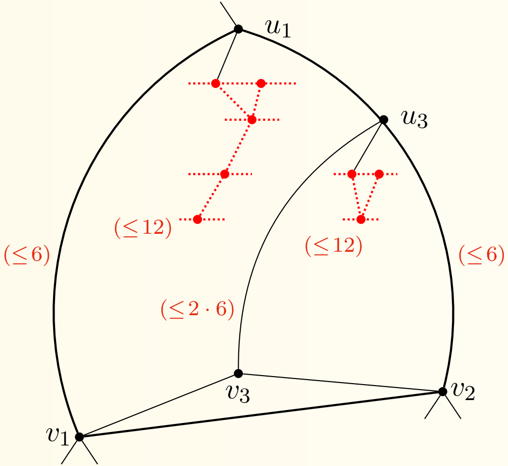
- Then contract, by the BFS layers inside, the recursively contracted vertices with those of vertical "divisor" P_3 down to one or two vert.



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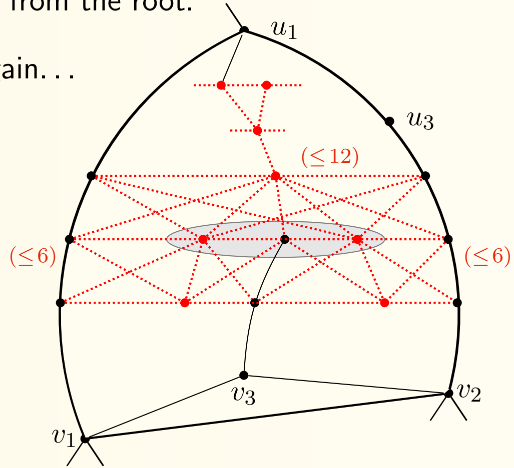
- Then contract, by the BFS layers inside, the recursively contracted vertices with those of vertical "divisor" P_3 down to one or two vert. Proceed in increasing distance from the root.
- And, check the red degrees again. . .

□



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- The key is to consider a *left-aligned BFS tree* instead of general one.

  This makes the left and right bounding paths non-symmetric, and their claimed recursive red degrees are ≤ 5 and ≤ 3, respectively.

- We do not contract the partial solutions of the subcases layer-by-layer, but first fully contract the right subcase with the dividing path P_3, and then the outcome with the left subcase.

4 Towards Proving Twin-Width ≤ 9

Several adjustments at different places are necessary (which make the proof quite technical)...

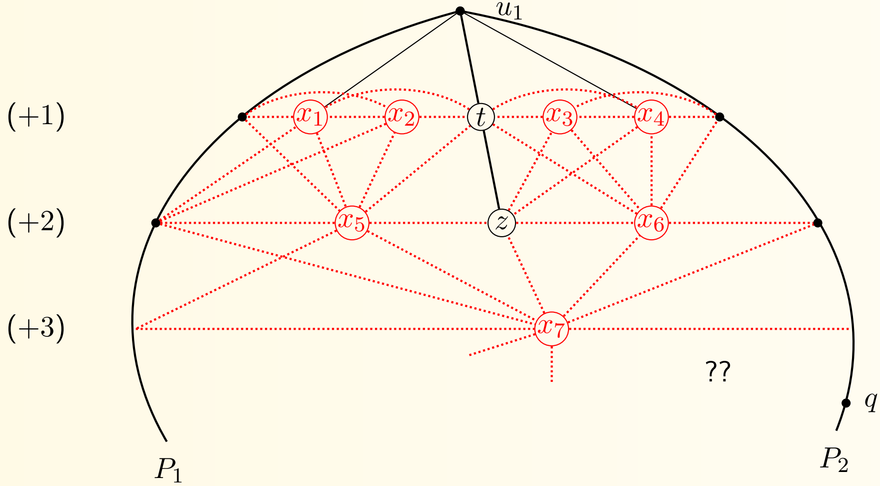
- The key is to consider a *left-aligned BFS tree* instead of general one.

  This makes the left and right bounding paths non-symmetric, and their claimed recursive red degrees are ≤ 5 and ≤ 3, respectively.

- We do not contract the partial solutions of the subcases layer-by-layer, but first fully contract the right subcase with the dividing path P_3, and then the outcome with the left subcase.

- Now we proceed from the farthest BFS layers towards the root, and a few of the layers closest to the sink are possibly handled ad-hoc.

Illustrating the Proof Adjustments...



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  - Take the dual of the *soccer ball graph*;
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  - Stepping further, inscribe a degree-3 vertex inside each face of the previous. The result seems to have twin-width ≥ 7, but a careful (computer asisted?) proof is needed.

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Thank you for your attention.