# 1 Transformation to normal forms

**Exercise 1.1:** Transform the following formulae to DNF using truth tables and using De Morgan's laws and transformations between logical connectives.

a) $(A \Rightarrow B) \Rightarrow C$

b) $(A \Leftrightarrow B) \vee \neg C$

c) $(A \Leftrightarrow B) \Rightarrow (C \vee D)$

*A formula is in disjunctive normal form (DNF) if it has the form $\alpha_1 \vee \ldots \vee \alpha_n$, where $\alpha_i = A_{i1} \wedge \ldots \wedge A_{ij_i}$ and every $A_{ij}$ is a propositional variable or its negation.*

**Exercise 1.2:** Transform the following formulae to CNF using truth tables and using De Morgan's laws and transformations between logical connectives.

a) $(A \Leftrightarrow B) \Rightarrow (\neg A \wedge C)$

b) $(A \Rightarrow B) \Leftrightarrow (A \Rightarrow C)$

*A formula is in conjunctive normal form (CNF) if it has the form $\alpha_1 \wedge \ldots \wedge \alpha_n$, where $\alpha_i = A_{i1} \vee \ldots \vee A_{ij_i}$ and every $A_{ij}$ is a propositional variable or its negation.*

**Exercise 1.3:** We know we can perform the following chemical reactions:

$$\begin{aligned} \mathrm{MgO} + \mathrm{H_2} &\rightarrow \mathrm{Mg} + \mathrm{H_2O} \\ \mathrm{C} + \mathrm{O_2} &\rightarrow \mathrm{CO_2} \\ \mathrm{CO_2} + \mathrm{H_2O} &\rightarrow \mathrm{H_2CO_3} \end{aligned}$$

We have the substances C, $H_2$, $O_2$ and MgO. Prove by resolution that $H_2CO_3$ is a logical consequence of the reactions.

**Exercise 1.4:** Transform the following formulae to PNF:

a) $\forall y(\exists x P(x,y) \Rightarrow Q(y,z)) \wedge \exists y(\forall x R(x,y) \vee Q(x,y))$

b) $\exists x R(x,y) \Leftrightarrow \forall y P(x,y)$

c) $(\forall x \exists y Q(x,y) \vee \exists x \forall y P(x,y)) \wedge \neg \exists x \exists y P(x,y)$

d) $\neg(\forall x \exists y P(x,y) \Rightarrow \exists x \exists y R(x,y)) \wedge \forall x(\neg \exists y Q(x,y))$

*A formula is in prenex normal form (PNF) if all quantifiers are at the beginning, i.e. it has the form form $Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \varphi$, where $Q_1, \ldots, Q_n \in \{\forall, \exists\}$ and $\varphi$ is a formula without quantifiers (an open formula).*

# 2 Skolemization and unification

**Exercise 2.1:** Perform a Skolemization of the following formulae in PNF:

a) $\forall y_1 \forall x_1 \exists y_2 \forall x_2[(\neg P(x_1,y_1) \vee Q(y_1,a)) \wedge (R(x_2,y_2) \vee Q(x_1,y_2))]$

b) $\forall x_1 \forall y_1 \exists y_2 \exists x_2[(\neg R(x_1,y_2) \vee P(b,y_1)) \wedge (\neg P(x_1,y_2) \vee R(x_2,b))]$

c) $\exists x_1 \forall y_1 \exists x_2(S(y_1) \vee R(x_1,x_2))$

**Exercise 2.2:** Find the most general unifiers of the following sets of literals:

a) $S = \{P(x, f(y), z), P(g(a), f(w), u), P(v, f(b), c)\}$

b) $T = \{Q(h(x, y), w), Q(h(g(v), a), f(v)), Q(h(g(v), a), f(b))\}$

**Exercise 2.3:** Find all possible resolvents of the following pairs of clauses:

a) $C_1 = \{P(x, y), P(y, z)\}, C_2 = \{\neg P(u, f(u))\}$

b) $C_1 = \{P(x, x), \neg R(x, f(x))\}, C_2 = \{R(x, y), Q(y, z)\}$

c) $C_1 = \{P(x, y), \neg P(x, x), Q(x, f(x), z)\}, C_2 = \{\neg Q(f(x), x, z), P(x, z)\}$

# 3 Resolution

**Exercise 3.1:** Prove the following logical consequences:

a) $\{\forall x P(x, x), \forall x \forall y \forall z((P(x, y) \wedge P(y, z)) \Rightarrow P(z, x))\} \models \forall x \forall y(P(x, y) \Rightarrow P(y, x))$

b) $\{\forall x \forall y \forall z((P(x, y) \wedge P(y, z)) \Rightarrow P(x, z)), \forall x \forall y(P(x, y) \Rightarrow P(y, x))\} \models \forall x \forall y \forall z((P(x, y) \wedge P(z, y)) \Rightarrow P(x, z))$

**Exercise 3.2:** Transform the following statements into formulae of predicate logic and prove the described consequences:

a) Assume that the following three statements hold:

- There exists a dragon (denote it $D/1$).
- Dragons sleep ($S/1$) or hunt ($L/1$).
- If a dragon is hungry ($H/1$), it cannot sleep.

Conclusion: *If a dragon is hungry, it hunts.*

b) Assume that the following two statements hold:

- All barbers ($B/1$) shave ($S/2$) everyone who does not shave himself.
- No barber shaves anyone who shaves himself.

Conclusion: *There are no barbers.*

**Exercise 3.3:** Refute the following set of clauses

$$\begin{aligned} S = & \{\{P(x), \neg Q(x, f(y)), \neg R(a)\}, \{R(x), \neg Q(x, y)\}, \{\neg P(x), \neg Q(y, z)\}, \\ & \{P(x), \neg R(x)\}, \{R(f(b))\}, \{Q(x, y), \neg P(y)\}\} \end{aligned}$$

using general resolution, linear resolution, LI resolution, LD resolution, and SLD resolution.

# 4 SLD-trees and resolution in Prolog

**Exercise 4.1:** Find a resolution refutation of the set of clauses given by the following program and question in Prolog.

```
1. r :- p, q.        5. t.
2. s :- p, q.        6. q.
3. v :- t, u.        7. u.
4. w :- v, s.        8. p.

?- w.
```

**Exercise 4.2:** Vytvořte SLD-strom pro následující program (`Program 1.`) a dotaz v Prologu a zjistěte, jak se projeví změna pořadí klauzulí (`Program 2.`) v definici programu na výsledné podobě SLD-stromu.

```
Program 1:                        Program 2:
 1. p :- q,r.      4. r :- q.      1. p :- r.       4. r.
 2. p :- r.        5. r.           2. p :- q,r.     5. r :- q.
 3. q :- p.                        3. q :- p.

?- p,q.
```

**Exercise 4.3:** Draw the SLD-trees for the following Prolog programs and goals:

```
Program 1:                        Program 2:
 1. p :- a,r.  5. r :- t,a.        1. p :- s,t.  5.  r :- w.
 2. a :- b.    6. r :- s.          2. p :- q.    6.  r.
 3. a.         7. s.               3. q.         7.  s.
 4. b :- a.                        4. q :- r.    8.  t :- w.

 ?- p.                             ?- p.
```

**Exercise 4.4:** Draw the SLD-tree for the following Prolog program and goal:

```
1. p(f,g).      3. p(Z,X) :- p(X,Y), p(Y,Z).
2. p(X,X).

?- p(Y,f).
```

**Exercise 4.5:** Draw the SLD-tree for the following Prolog program and goal:

```
1. p(X,Y) :- q(X,Z), r(Z,Y).        7. s(X) :- t(X,a).
2. p(X,X) :- s(X).                  8. s(X) :- t(X,b).
3. q(X,b).                          9. s(X) :- t(X,X).
4. q(b,a).                         10. t(a,b).
5. q(X,a) :- r(a,X).               11. t(b,a).
6. r(b,a).

?- p(X,X).
```

**Exercise 4.6:** Find an SLD-resolution refutation of the goal `?- reverse([a,b,c],X).` assuming that the predicate `reverse/2` is defined in the following way:

```
reverse(L1,L2) :- rev(L1,[],L2).
rev([H|T],A,L) :- rev(T,[H|A],L).
rev([],L,L).
```

# 5 Tableaux in propositional logic

**Exercise 5.1:** Draw a finished tableau with the root

$$F(((C \lor d) \land (D \lor \neg d)) \Leftrightarrow (C \lor D)),$$

where $C, D, d$ are propositional letters.

**Exercise 5.2:** Using tableaux prove that the following formulae are tautologies:

a) $\neg(p \Rightarrow q) \;\Rightarrow\; (q \Rightarrow p)$

b) $((p \lor q) \Rightarrow (p \lor r)) \;\Rightarrow\; (p \lor (q \Rightarrow r))$

**Exercise 5.3:** Prove the following logical consequence:

$$\{q \Rightarrow r, r \Rightarrow (p \land q), p \Rightarrow (q \lor r)\} \models (p \Leftrightarrow q).$$

# 6 Tableaux in predicate logic

**Exercise 6.1:** Using tableaux prove that the following formulae are tautologies:

a) $\Phi_1 \equiv \forall x \varphi(x) \Rightarrow \exists x \varphi(x)$

b) $\Phi_2 \equiv \forall x(P(x) \Rightarrow Q(x)) \Rightarrow (\forall x P(x) \Rightarrow \forall x Q(x))$

c) $\Phi_3 \equiv \forall x(\varphi(x) \land \psi(x)) \Leftrightarrow (\forall x \varphi(x) \land \forall x \psi(x))$

d) $\Phi_4 \equiv \exists y \forall x(P(x,y) \Leftrightarrow P(x,x)) \Rightarrow \neg\forall x \exists y \forall z(P(z,y) \Leftrightarrow \neg P(z,x))$

**Exercise 6.2:** Prove that the formula $\forall x P(x)$ is a logical consequence of the following formulae:

$$\forall x((Q(x) \lor R(x)) \Rightarrow \neg S(x))$$
$$\forall x((R(x) \Rightarrow \neg P(x)) \Rightarrow (Q(x) \land S(x)))$$

**Exercise 6.3:** Prove the following logical consequences using tableaux.

a) Assume that the following three statements hold:

- There exists a dragon (denote it $D/1$).
- Dragons sleep ($S/1$) or hunt ($L/1$).
- If a dragon is hungry ($H/1$), it cannot sleep.

Conclusion: *If a dragon is hungry, it hunts.*

b) Assume that the following two statements hold:

- All barbers ($B/1$) shave ($S/2$) everyone who does not shave himself.
- No barber shaves anyone who shaves himself.

Conclusion: *There are no barbers.*

# 7 Tableaux in modal logic

**Exercise 7.1:** Using tableaux prove that the following formulae are tautologies.

a) $\Phi_1 \equiv (\Box\forall x\varphi(x)) \Rightarrow (\forall x\Box\varphi(x))$

b) $\Phi_2 \equiv (\Box(\varphi \Rightarrow \psi)) \Rightarrow ((\Box\varphi \Rightarrow \Box\psi))$

c) $\Phi_3 \equiv \neg\Diamond(\neg(\varphi \wedge \exists x\psi(x)) \wedge \exists x(\varphi \wedge \psi(x)))$, $x$ is not free in the formula $\varphi$

d) $\Phi_4 \equiv \Diamond\exists x(\varphi(x) \Rightarrow \Box\psi) \Rightarrow \Diamond(\forall x\varphi(x) \Rightarrow \Box\psi)$, $x$ is not free in the formula $\psi$

**Exercise 7.2:** Consider the tableau with the root $Fw \Vdash \forall x\Box\varphi(x) \Rightarrow \Box\forall x\varphi(x)$ given in Figure 1. Decide whether the tableau is correct and explain your decision.

**Exercise 7.3:** Prove the following logical consequences:

a) $\{\varphi\} \models \Box\varphi$

b) $\{\forall x\varphi(x)\} \models \Box\forall x\varphi(x)$

c) $\{\forall x\varphi(x)\} \models \forall x\Box\varphi(x)$

d) $\{\varphi \Rightarrow \Box\varphi\} \models \Box\varphi \Rightarrow \Box\Box\varphi$

$$Fw \Vdash \forall x \Box \varphi(x) \Rightarrow \Box \forall x \varphi(x)$$
$$|$$
$$Tw \Vdash \forall x \Box \varphi(x)$$
$$|$$
$$Fw \Vdash \Box \forall x \varphi(x)$$
$$|$$
$$wSv$$
$$|$$
$$Fv \Vdash \forall x \varphi(x)$$
$$|$$
$$Fv \Vdash \varphi(c)$$
$$|$$
$$Tw \Vdash \forall x \Box \varphi(x)^*$$
$$|$$
$$Tw \Vdash \Box \varphi(c)$$
$$|$$
$$Tw \Vdash \Box \varphi(c)$$
$$|$$
$$Tv \Vdash \varphi(c)$$
$$|$$
$$\otimes$$

Figure 1:

# 8 Inductive logic programming

**Exercise 8.1:** Consider the propositional letters $a, b, c, d, e$ and the table

| $a$ | $b$ | $c$ | $d$ | $e$ | *class* |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | + |
| 1 | 1 | 0 | 1 | 1 | + |
| 1 | 1 | 0 | 1 | 0 | − |

$+$ = positive examples
$-$ = negative examples

Find all specialisations of the propositional formula $b$. Indicate which of them cover the negative example.

**Exercise 8.2:** Draw a part of specialisation graph with the root `niece(X,Y)` containing the clause:

```
niece(X,Y) :- female(X), sibling(Y,Z), parent(Z,X).
```