Randomness extractors

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Part I

Extracting randomness

Random Numbers in Computer Science

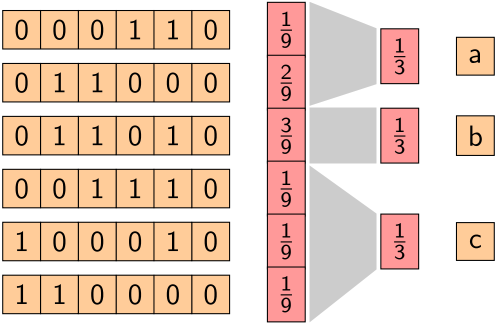
- Random numbers are of crucial importance for a waste number of computer science applications.
- Cryptography is impossible without random numbers.
  - Cryptographic keys - encryption, authentication, digital signatures
  - Random choices in cryptographic algorithms and protocols - zero knowledge proofs
- Randomized algorithms
- Communication protocols

Practically all these applications

- inherently require randomness generated uniformly
- or their analysis is performed for uniform random numbers.

Extraction from Know Probability Distribution

- In contrast to our requirements, most available sources of randomness generate non-uniform output.
- We have to partition the set of outputs into set of constant probability.
- Depending on the output probability distribution, this may be impossible.



Extraction from Unknown Probability Distribution

- The probability distribution of the random number generator output may vary during the computation.
- This might be due to
  - low quality of the generator design,
  - external hard-to-control effects, such as temperature,
  - or an attack of an adversary.
- Extraction is still possible, given some limitations on the output probability distributions.

Von Neumann Extractor

- Source produces a sequence of random bits, that are generated independently according to (an unknown) a fixed probability distribution.
- On each position the source generates independently

$$0 \text{ with probability } p$$
$$1 \text{ with probability } (1 - p).$$

- Von Neumann extractor divides the bit sequence into pairs and for each pair of bits it takes action depending on the value

00 outputs nothing

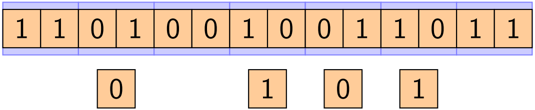
11 outputs nothing

01 outputs 0

11 outputs 1.

Von Neumann Extractor

- For the aforementioned source the output is always a sequence of independent and uniformly distributed bits.



Part II

Randomness Extractors

Towards extractor definition

- The purpose of an extractor is to transform an input (biased) probability distribution to a probability distribution that is (close to) uniform distribution.
- Assume we have a biased distribution X on $\mathbf{X}$.
- A randomness extractor is function $e : \mathbf{X} \rightarrow \mathbf{Y}$, such that the distribution Y on $\mathbf{Y}$ induced by the distribution X, i.e.

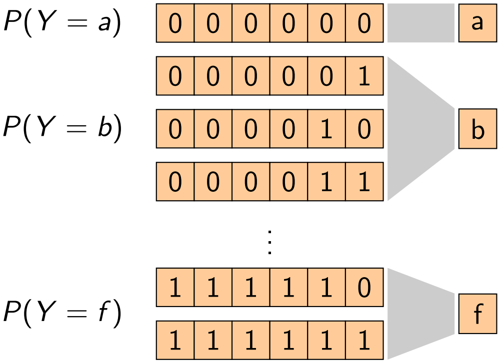
$$P(Y = y) = \sum_{x \in \mathbf{X}, e(x) = y} P(X = x),$$

  is close (to be specified later) to the uniform distribution.
- Such an extractor has natural limitations, namely for a fixed e, and two distributions X_1 and X_2 mapped by e to uniform distribution, it holds that

$$\forall\, y \in \mathbf{Y} \sum_{x \in \mathbf{X}, e(x) = y} P(X_1 = x) = P(Y = y) = \sum_{x \in \mathbf{X}, e(x) = y} P(X_2 = x).$$

This means that e partitions $\mathbf{X}$ to pre-images of elements of $\mathbf{Y}$.

$$e : \mathbf{X} \rightarrow \mathbf{Y}$$



Towards extractor definition

- We may overcome this limitation by allowing a (small) auxiliary uniform input Z.
- This would give us the seeded extractor $e : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$.
- We naturally expect that the extractor should be useful, i.e. to produce some randomness. This is rephrased as $|Y| > |Z|$.

Trace Distance of Probability Distributions

Definition

Let X and Y be random variables defined on the same sample space $\mathcal{S}$ with probability distributions p_X and p_Y, respectively. The **trace distance** (or L_1 **distance**) of random variables X and Y is

$$d(X,Y) = \frac{1}{2}\sum_{a\in\mathcal{S}} |p_X(a) - p_Y(a)| = \max_{A\subseteq\mathcal{S}} |P(X\in A) - P(Y\in A)|. \quad (1)$$

X and Y are ϵ-**close in** L_1 iff

$$d(X,Y) \leq \epsilon. \quad (2)$$

Extractor Definition

Definition

Let $\mathcal{P}(\mathbf{X})$ be the set of all probability distributions on $\mathbf{X}$, and $\mathcal{S} \subset \mathcal{P}(\mathbf{X})$. Then $e : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ is a $(\mathcal{S}, \epsilon)$ (seeded) **randomness extractor** iff for all $X \in \mathcal{S}$

$$d(e(X, U_Z), U_Y) \leq \epsilon, \tag{3}$$

where U_Z is the uniform distribution on $\mathbf{Z}$ and U_Y is the uniform distribution on $\mathbf{Y}$.

Part III

Min-entropy

Randomness Extractor and min-entropy

Definition

The **min–entropy** of a probability distribution X is

$$H_\infty(X) = \min_{x \in \mathbf{X}} -\log P(X = x) = -\log \max_{x \in \mathbf{X}} P(X = x). \tag{4}$$

It is a good measure of the amount of randomness contained in the input probability distribution, as demonstrated by the next theorem.

Requirements for Source of Randomness

Theorem

Let X be a random variable with image $\mathbf{X} = \{0,1\}^n$ satisfying $H_\infty(X) \leq k-1$ for some $k \in \mathbb{N}$. Then there no $(\{X\}, 0)$ extractor e with $\mathbf{Z} = \{0,1\}^d$ and $\mathbf{Y} = \{0,1\}^m$ such that $m \geq k+d$.

Proof.

The fact that $H_\infty(X) \leq k-1$ implies that there is some element x such that $P(X=x) \geq 2^{-(k-1)}$. Therefore, for any auxiliary input $z \in \mathbf{Z}$ the probability of the corresponding output $e(x,z)$ is at least $2^{-(k-1)}2^{-d}d = 2^{-(k+d-1)} > 2^{-m}$ and therefore the output probability distribution is not uniform and its distance from the uniform distribution is bounded by the min-entropy of the input. □

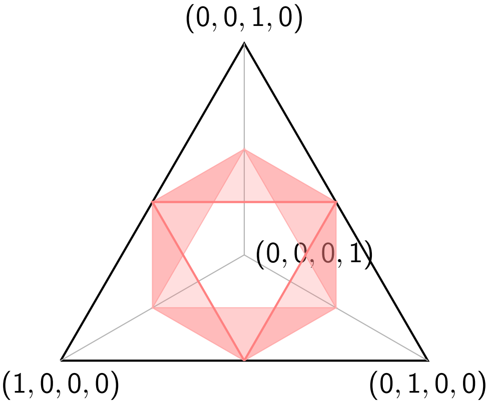
Previous theorem shows us that the gain of randomness extraction is limited by the min–entropy of the source distribution.

Part IV

Sources of Randomness

Min-Entropy Source

A first example of an extractable set of probability distributions is the min-entropy source. We define the source with min-entropy k as $\mathcal{S} \subset \mathcal{P}(\mathbf{X})$ such that $\forall X \in \mathcal{S}\ H_\infty(X) \geq k$.



Min-Entropy Extractor

Definition

The function $e : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ is a (k, ϵ) (seeded) **randomness extractor** iff for all X with $H_\infty(X)$ it holds that

$$d(e(X, U_Z), U_Y) \leq \epsilon, \quad (5)$$

where U_Z is the uniform distribution on $\mathbf{Z}$ and U_Y is the uniform distribution on $\mathbf{Y}$.

Towards Definition of Extractor

Theorem

There is no function $e : \{0,1\}^n \to \{0,1\}$ *giving a single random bit (uniform distribution on* $\{0,1\}$*) as an output for any input random variable* X *on* n*–bit strings satisfying* $H_\infty(X) \geq n-1$*.*

Intuitively, an input distribution with min–entropy at least $n-1$ contains much more randomness than necessary to obtain a single random bit.

Proof.

For every function e there is a bit $b \in \{0,1\}$ such that $|\{x \in \{0,1\}^n | e(x) = b\}| \geq 2^{n-1}$ since there are 2^n inputs in the domain of e. Let us consider a random variable X uniformly distributed on the set $\{x \in \{0,1\}^n | e(x) = b\} \subset \{0,1\}^n$. Such a random variable obeys $H_\infty(X) \geq n-1$ and yet the output distribution $e(X)$ is constant, i.e. $P(e(X) = b) = 1$. □

Part V

Extractors for Min-entropy Sources

Min-Entropy Strong Extractor

Definition

The function $e : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ is a (k, ϵ) (seeded) **strong randomness extractor** iff for all X with $H_\infty(X)$ it holds that

$$d\big([U_z, e(X, U_Z)], [U_Z, U_Y]\big) \leq \epsilon, \quad (6)$$

where U_Z is the uniform distribution on $\mathbf{Z}$ and U_Y is the uniform distribution on $\mathbf{Y}$.

- The advantage of the strong extractor is that the output is close to the uniform distribution even if the value of U_Z is known.
- Next we will show how to implement a strong extractor using Wegman-Carter hashing.

Min-Entropy Extractor

Theorem

Let X *be a random variable defined on* $\mathbf{X} = \{0,1\}^n$ *with min-entropy* $H_\infty(X) \geq k$, $H = \{h | h : \{0,1\}^n \rightarrow \{0,1\}^{k-2e}\}$ *be a universal*$_2$ *class of hash functions. Let* $x \in_R \mathbf{X}$ *be randomly chosen from* $\mathbf{X}$ *according to* X *and* h *be randomly and uniformly chosen from* H*. Then the distribution of* $(h, h(x))$ *is* 2^{-e} *close to the uniform distribution in the trace distance, i.e. application of a function randomly chosen from* H *is a* $(k, 2^{-e})$ *strong randomness extractor.*

Min-Entropy Extractor

Theorem

Let $X_1, X_2, \ldots, X_l$ be independent identically distributed random variables each defined on $\mathbf{X} = \{0,1\}^n$ with min-entropy $H_\infty(X) \geq k$, $H = \{h | h : \{0,1\}^n \to \{0,1\}^{k-2e}\}$ be a universal$_2$ class of hash functions. Let $x_i \in_R \mathbf{X}$ be randomly chosen from $\mathbf{X}$ according to X_i and h be randomly and uniformly chosen from H. Then the distribution of $(h, h(x_1), \ldots, h(x_l))$ is $l2^{-e}$ close to the uniform distribution in the trace distance, i.e. l repeated applications of a fixed function randomly chosen from H is a $(k, l2^{-e})$ strong randomness extractor.