Shrub-Depth

a successful depth measure for dense graphs

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Ingredients: joint results with

J. Gajarský, R. Ganian, O. Kwon, J. Nešetřil, J. Obdržálek, S. Ordyniak, P. Ossona de Mendez

Measuring Width or Depth?

- **Being close to a TREE – "•-width"**

SPARSE

tree-width / branch-width
– showing a *structure*

DENSE

clique-width / rank-width
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- **Being close to a STAR – "•-depth"**

SPARSE

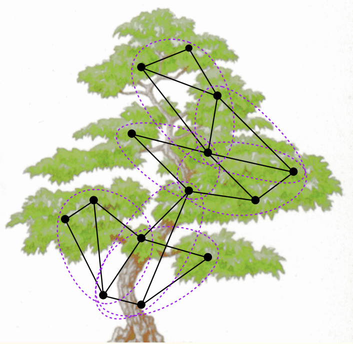
tree-depth
– containment in a *structure*

DENSE

???
(will show)

1 Recall: Width Measures

Tree-width $\boldsymbol{tw(G) \leq k}$ if whole G can be covered by bags of size $\leq k+1$, arranged in a "tree-like fashion".



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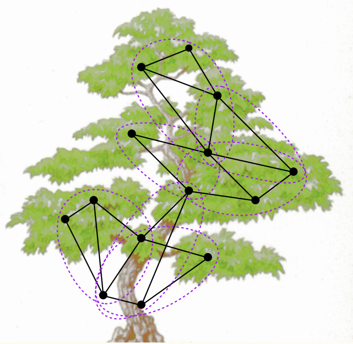

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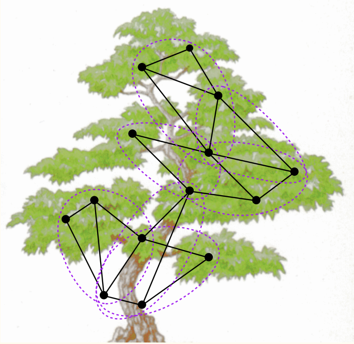

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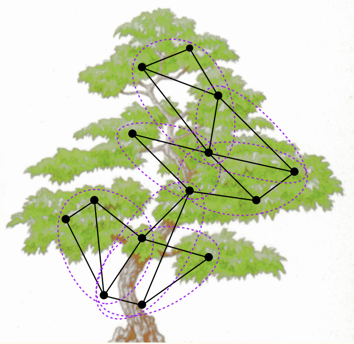

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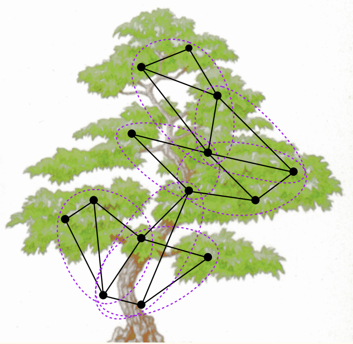

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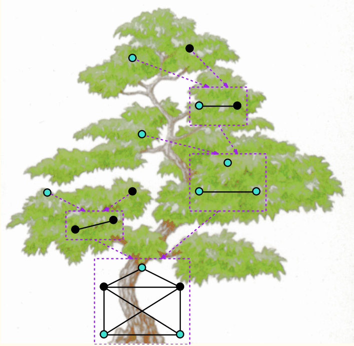
- asymptotically equivalent to NO large grid minor.

Clique-width $cwd(G) \leq k$ if G given by a k-*expression* (over k-labelled gr.),

k-expression $\sim$ disjoint unions, relabelling, edge-add. between labels.



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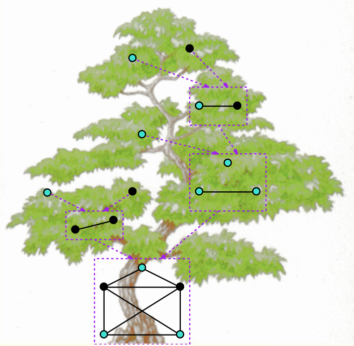

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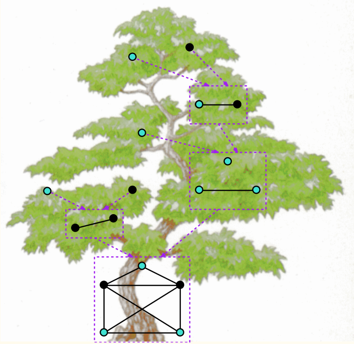

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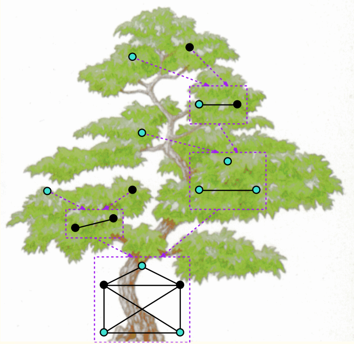

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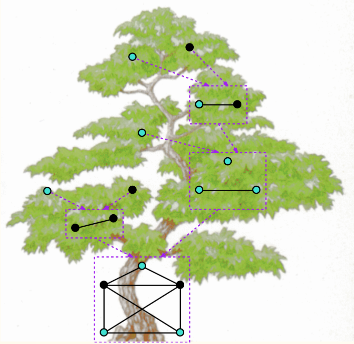

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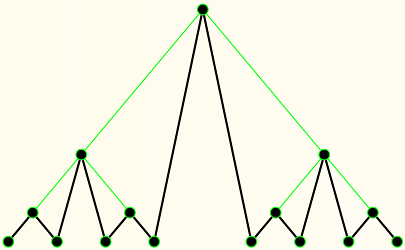
- no nice "excluded something" characterization known so far,
- but can be characterized by MSO_1 interpretations into trees.

and Depth Measures

Tree-depth $td(G) \le k$ if whole G is contained in the closure of a rooted forest of height $\le k+1$.



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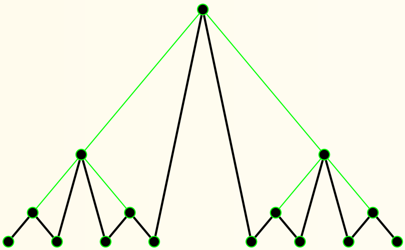

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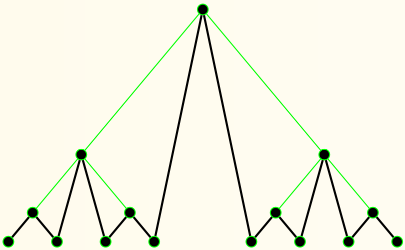

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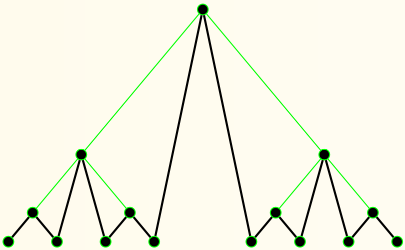

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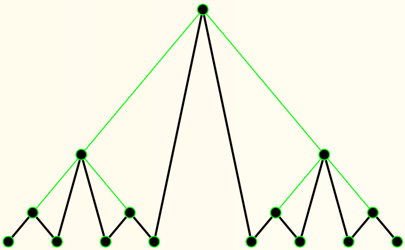

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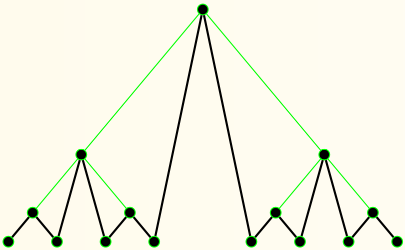

Structural properties

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- equiv. also to bounding the height of a tree-decomposition,
- asymptotically equivalent to a no long path subgraph,
- and well-behaved wrt. MSO_2 interpretations.

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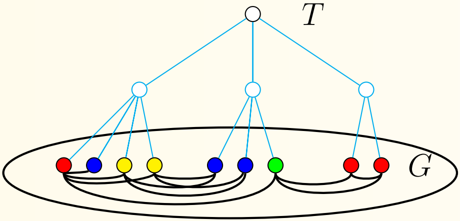
- Small also on some dense graphs, stable under complements.
  Recursive construction with limited inform. and of small depth.
- Stable under FO / MSO_1 interpretations!

2 Tree-models and Shrub-depth

Tree-model of m colours and depth d:

- a rooted tree T of height d,



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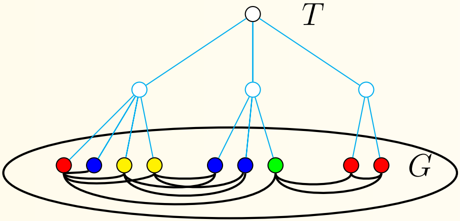

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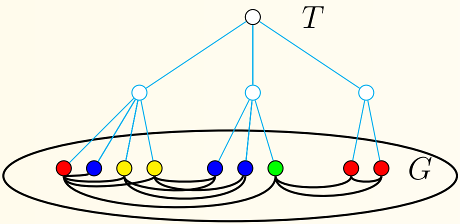

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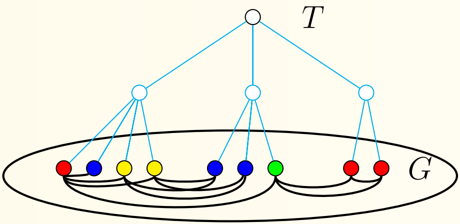

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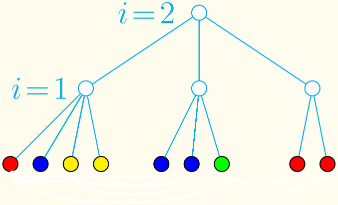

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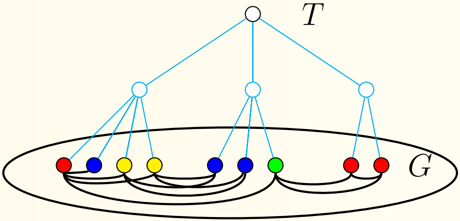

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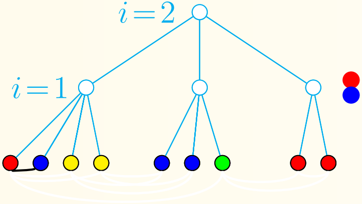

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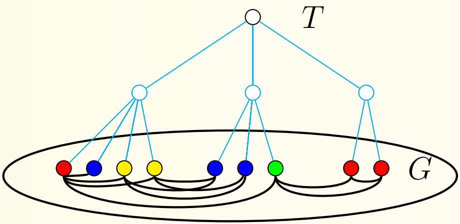

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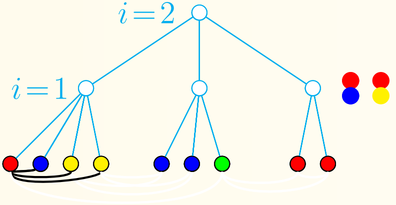

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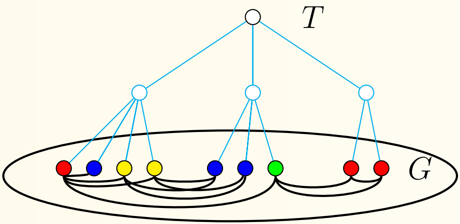

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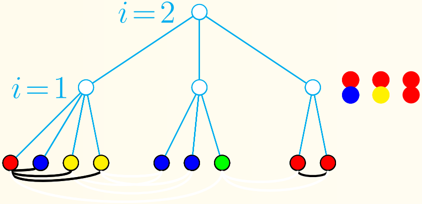

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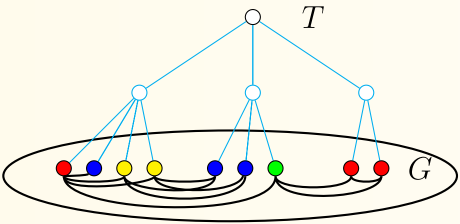

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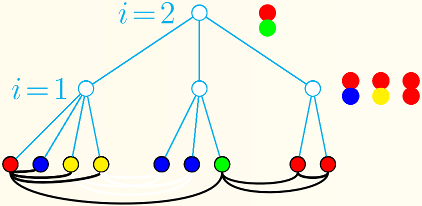

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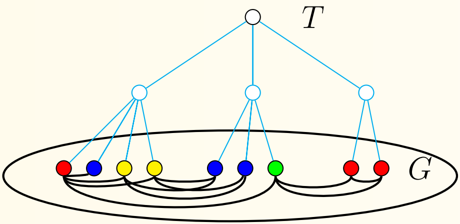

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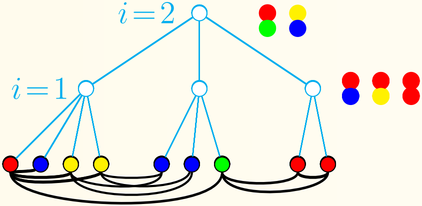

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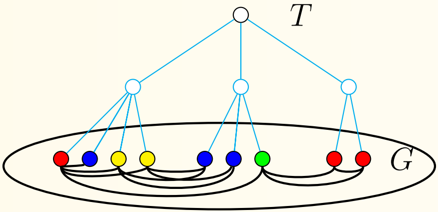

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$\{$ graphs with a tree-model of m *colours and depth* d $\}$

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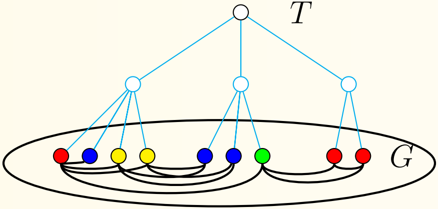

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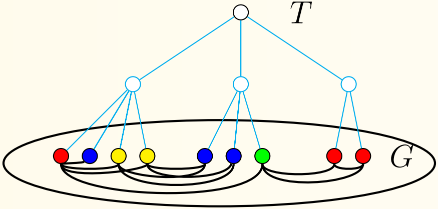

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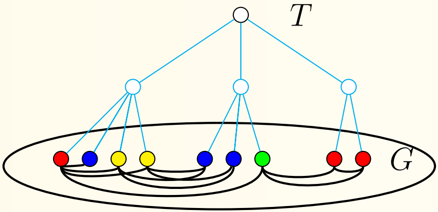

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there exists m such that $\mathcal{G} \subseteq \mathcal{TM}_m(d)$ (same m for all $\mathcal{G}$!),

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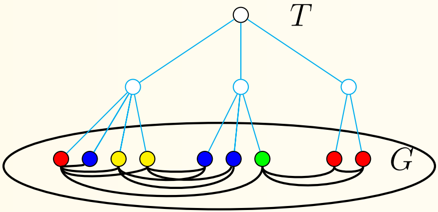

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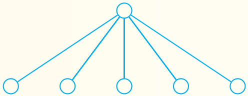
A graph class $\mathcal{G}$ is of **shrub-depth d** iff

there exists m such that $\mathcal{G} \subseteq \mathcal{TM}_m(d)$ (same m for all $\mathcal{G}$!),

while for all m' we have $\mathcal{G} \not\subseteq \mathcal{TM}_{m'}(d-1)$.

Examples and simple properties

- Shrub-depth 1: e.g., cliques, stars, ...
  gen. BND – bounded neighbourhood diversity.



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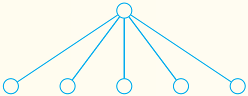

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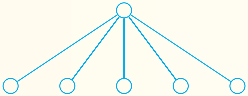

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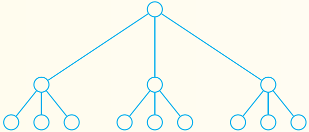

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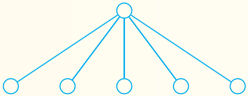

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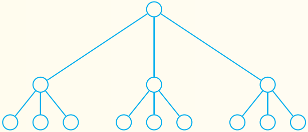

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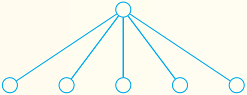

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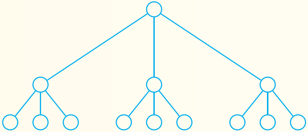

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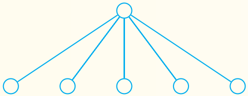

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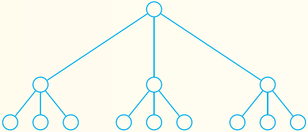

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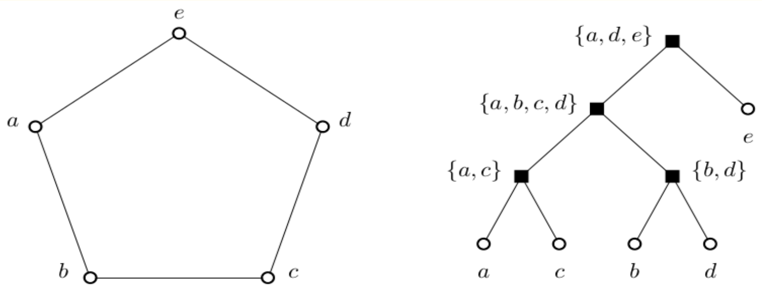

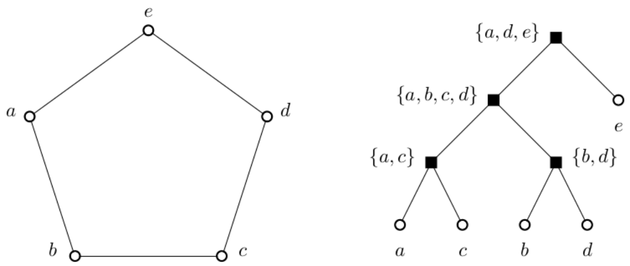

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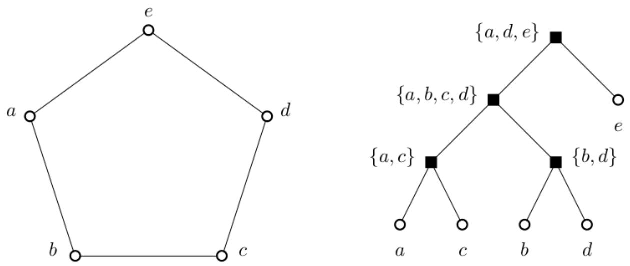

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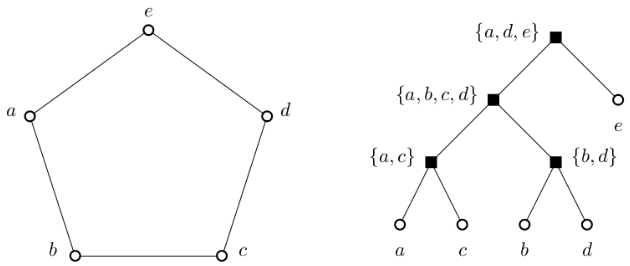

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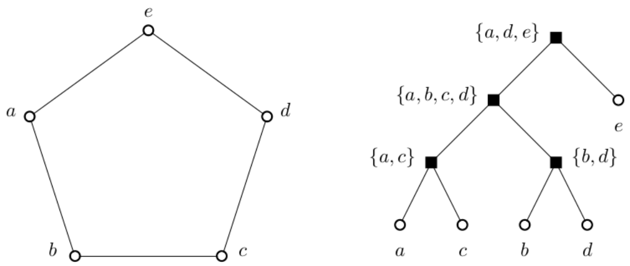

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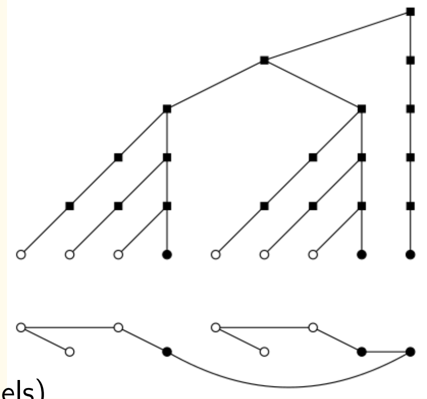
– start from an SC-depth tree, and "simulate" subset complem. via extra vert. with local complem.



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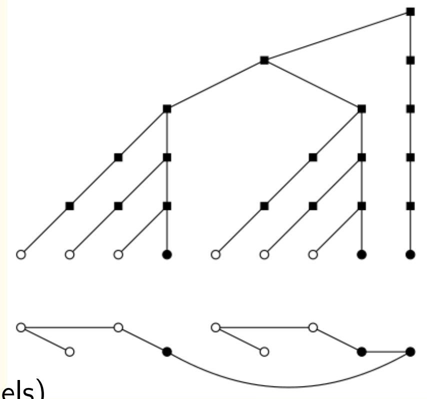

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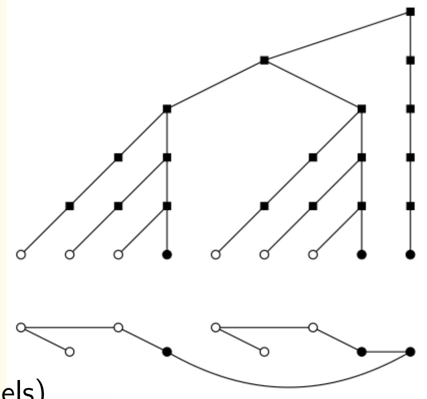

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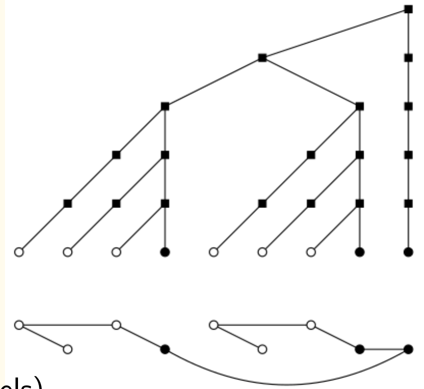

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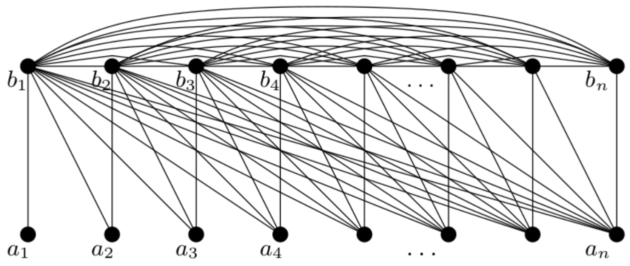

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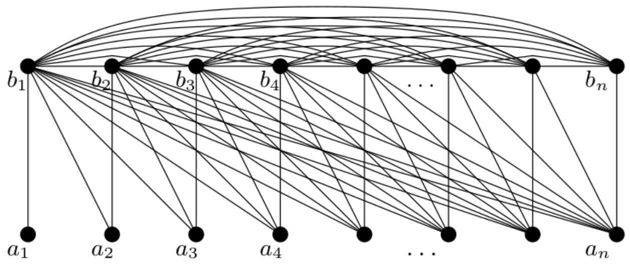

Conjecture A class $\mathcal{G}$ is of bounded shrub-depth

$$\iff$$

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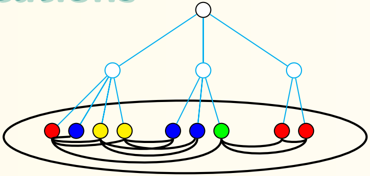

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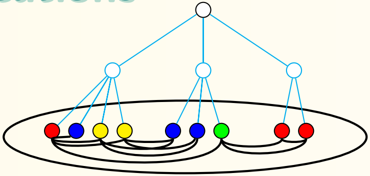

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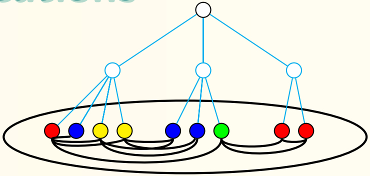

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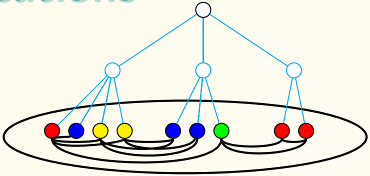

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  Moreover, doing this carefully, there is such universal $T \rightsquigarrow T_0$ to which $L(u'), L(v')$ can be added afterwards!

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Thank you for your attention.