Planar Graph Emulators

– Beyond Planarity in the Plane

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including results obtained with

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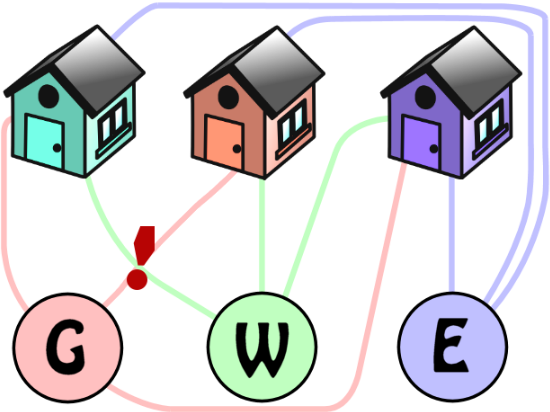
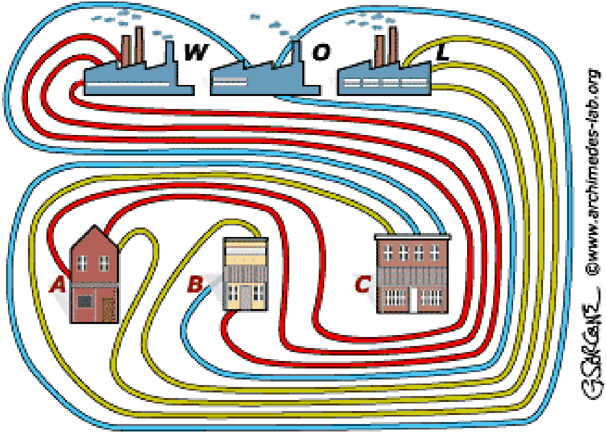
0 Planarity and Beyond

As everybody knows...



Can this crossing be avoided? NO?

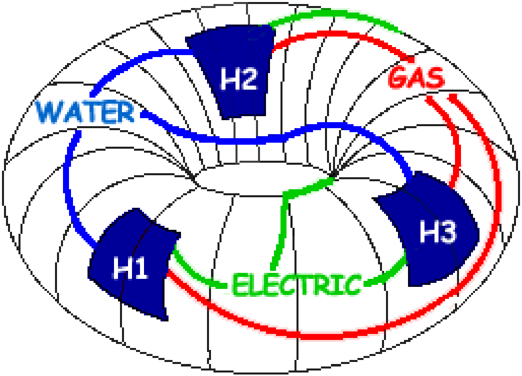
Let us try. . . ?



Oh NO!

This is nasty cheating.

One more try. . .

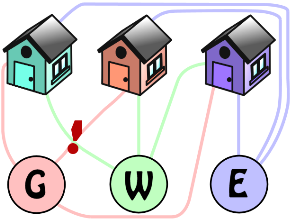


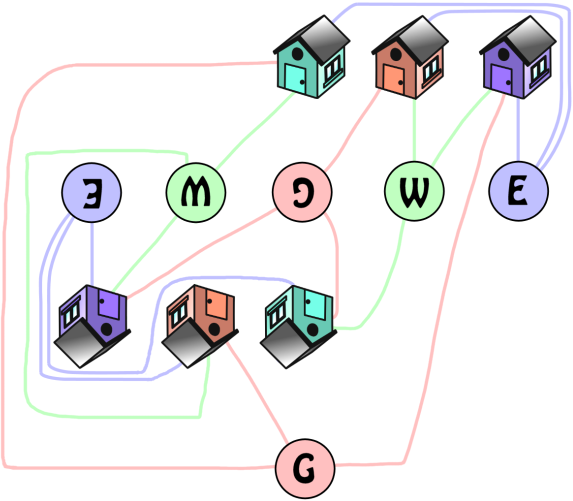
Yes, it works, somehow.

Not bad but not good either. . . We would like to stay in the plane!

Yet, a small miracle can happen. . .

– turning nonplanar into planar!





Origins and brief overview

- The question originated in mid-80's.

  Actually; two similar and independently discovered notions. . .

- **Planar covers** by [Seyia Negami]

  - the more restrictive notion of the two,
  - originally investigated in connection with flexibility of projective embeddings of 3-connected graphs.

- **Planar emulators** by [Michael Fellows]

  - the less restrictive notion of the two,
  - inv. in connection with modeling of graphs in other graphs.

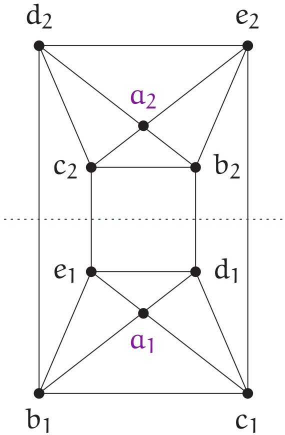
- Full definitions to follow. . .

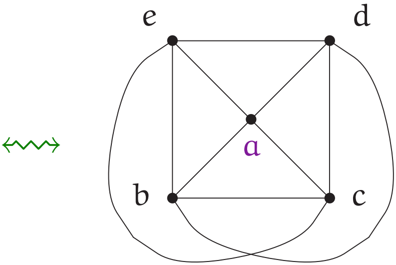
1 Planar Covers of Graphs

Motivation: Represent a nonplanar graph G *by planar* H *such that; exploring the two graphs locally, we cannot see any difference...*

- Having seen this for $K_{3,3}$, what about K_5—the other obstruction?

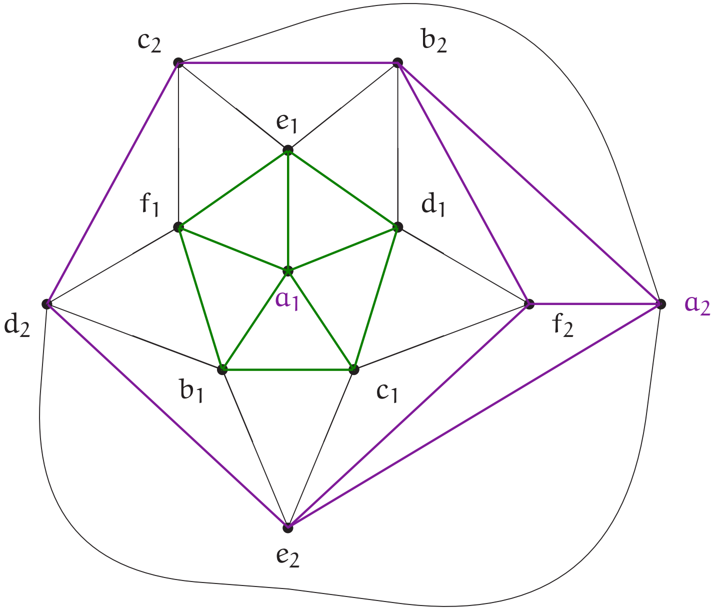
pl. H





$G = K_5$

Even more: Planar covering K_6



Limits of planar covering

- The covering relation preserves degrees of correspondent vertices.
- The multiplicity of covering vertices for each covered one is the same; we speak about a k-*fold cover* (*double covers* in the prev. examples).
- The complete graph K_7 has no finite planar cover:
  - the cover would have to have all vertex degrees 6, but a planar graph must contain a vertex of degree ≤ 5.
- Likewise, the complete bipart. graph $K_{4,4}$ has no finite planar cover:
  - the cover would have to have all vertex degrees 4, but a planar triangle-free graph must contain a vertex of degree ≤ 3.

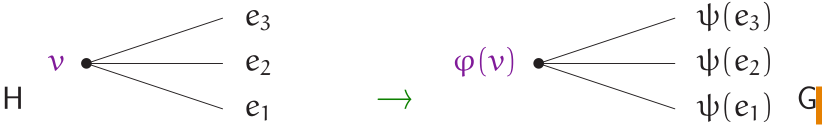
Formal definition

A graph H is a *cover* of a graph G if there exists a pair of onto mappings

(a *projection*) $\varphi : V(H) \to V(G), \qquad \psi : E(H) \to E(G)$

such that ψ maps the edges incident with each vertex v in H

bijectively onto the edges incident with $\varphi(v)$ in G.



We speak about a *planar cover* if H is a finite planar graph.

Remark. The edge $\psi(uv)$ has always ends $\varphi(u), \varphi(v)$, and hence only

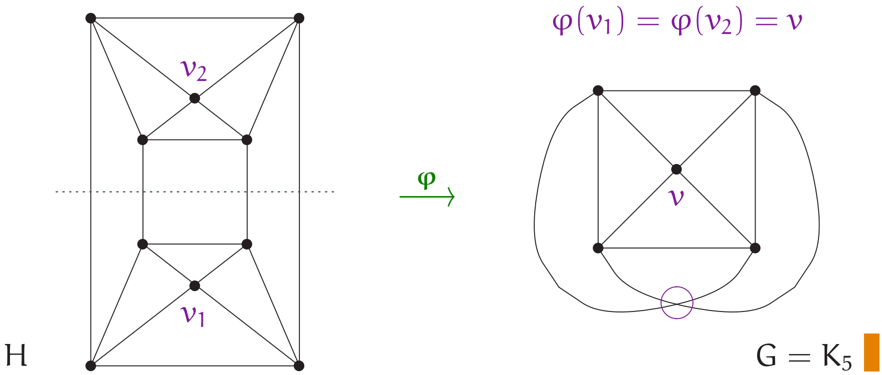
$\varphi : V(H) \to V(G)$, the *vertex projection*,

is enough to be specified for simple graphs

(φ is then a *locally bijective homomorphism*).

Back to examples of covering

- Revisiting a planar cover of K_5:



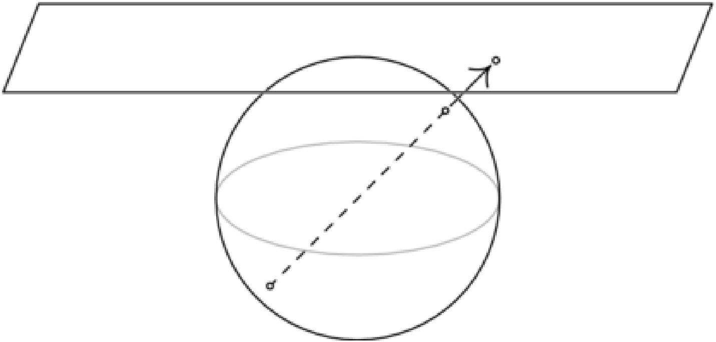
- **In general;**

  any graph embedded in the *projective plane* has a double planar cover,

  via the universal covering map from the sphere onto the proj. plane.

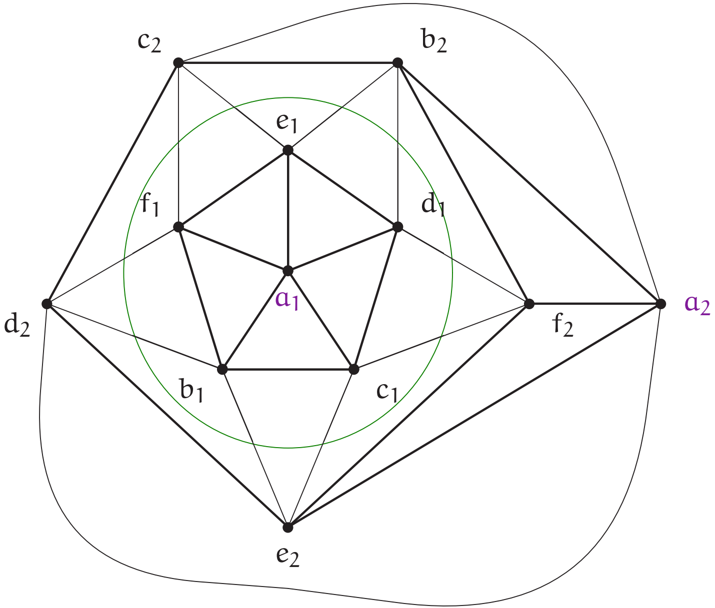
The projective plane

- The *projective plane* can be defined from the sphere as follows:



  - the points are the antipodal pairs (of points) of the sphere;
  - this "works" as the usual *Eucl. plane*, except at the equator.
- Can see the proj. plane as the usual plane with the "*line at infinity*";
- or, topologically, as the plane with a special region of a "*crosscap*".

Projective to double cover



2 Negami's Planar Cover Conjecture

Theorem 1 (Negami, 1986)

A connected graph has a double planar cover

$\iff$ *it embeds in the projective plane.*

Conjecture 2 (Negami, 1988)

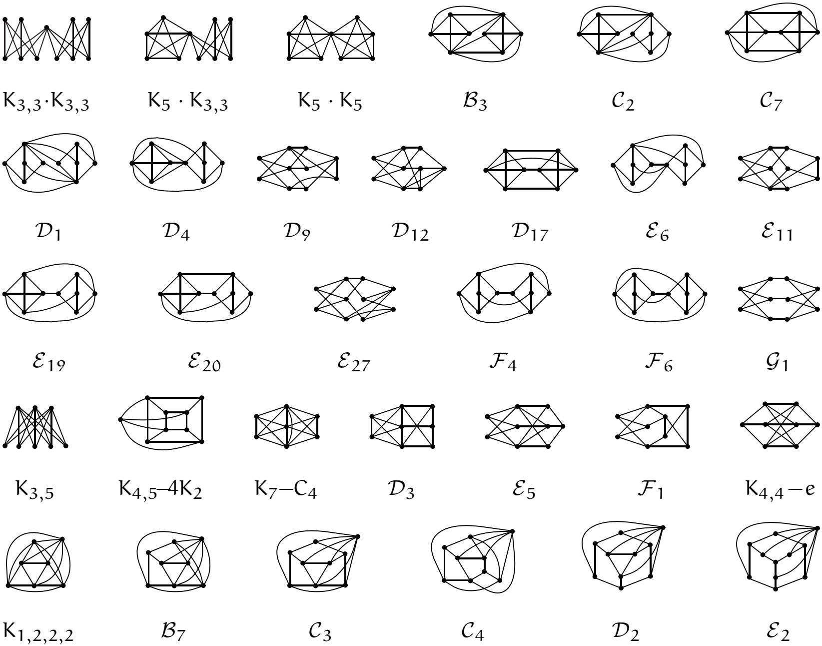
A connected graph has a finite planar cover

$\iff$ *it embeds in the projective plane.*

How to approach Negami's conjecture?

- The direction "projective $\rightarrow$ double pl. cover" is already known.
- We have to prove "not projective $\rightarrow$ no finite pl. cover"!
- For the latter, we have got a "*Kuratowski thm. for the projective plane*" [Archdeacon, 1981]... $\rightarrow$ we can test the obstructions.

The 32 conn. obstructions for the proj. plane



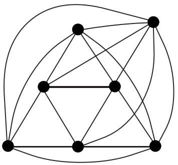
Known partial results

Around 10-years early development in Negami's conjecture led to...

Theorem 3 (Archdeacon, Fellows, Negami, PH; till 1998)
The following graphs cannot have finite planar covers:

- *the first* 19 *of the proj. obstructions ("two disjoint k-graphs"),*
- *the graph* $K_{3,5}$,
- *the graphs* $K_{4,5}-4K_2$, *and* K_7-C_4 *with its "YΔ family",*
- *the graph* $K_{4,4}-e$.

Corollary 4 *If one proved that the graph* $K_{1,2,2,2}$ *had no finite planar cover, then Negami's conjecture would be proved as well.*

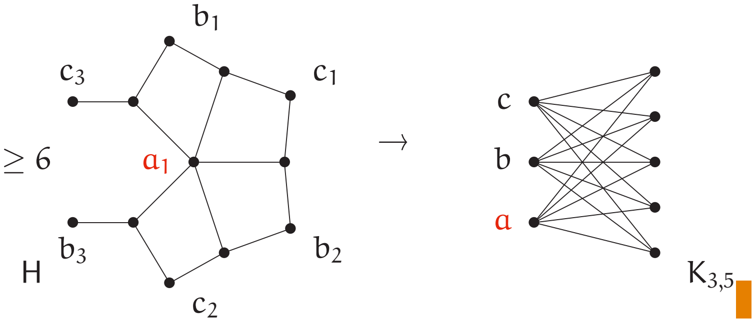


$K_{1,2,2,2}$

Sample proof: $K_{3,5}$

Theorem 5 (?? 1988, 1993) *The graph $K_{3,5}$ has no finite planar cover.*

Proof sketch. Assuming H is an n-fold planar cover of $K_{3,5}$, we shall derive a contradiction to Euler's formula ($\#vert. + \#faces - \#edges = 2$)...



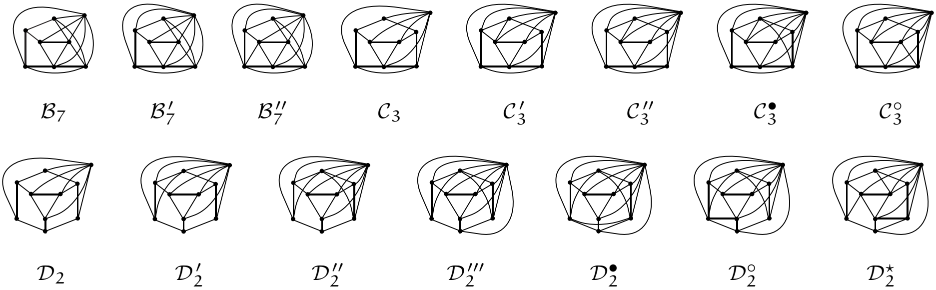
On the one hand, $\#faces = 2 + 15n - 8n = 7n + 2$.

On the other hand, a_1 in the pict. (and simil. each of b_i, c_i) accounts for $\leq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = 2 + \frac{1}{3}$ faces, which is altog. $\leq 3n \cdot (2 + \frac{1}{3}) \leq 7n$ faces in H, a contradiction. □

Current state of the art

For the last 15 years research has stalled, since the following finding:

Theorem 6 (Thomas and PH 1999, 2004)
If a connected graph G *has a finite planar cover but no projective embedding, then* G *is a planar expansion of* $K_{1,2,2,2}$ *or some graph from:*

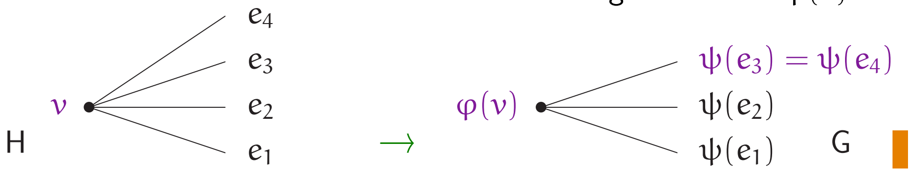


Corollary 7 *Negami's conjecture holds true for cubic graphs.*

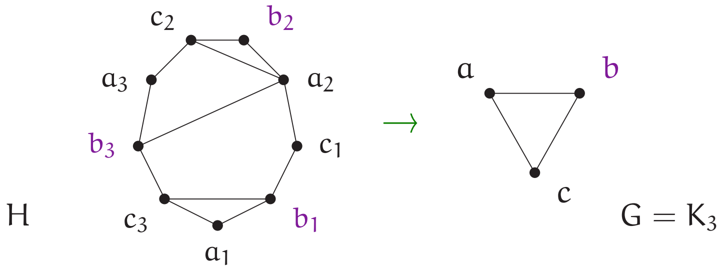
3 Planar Emulators: more relaxed

- $\varphi : V(H) \to V(G)$, an *emulator* vs. a cover:

  …map the edges inc. with v in H surjectively
  onto the edges inc. with $\varphi(v)$ in G.



- A nontrivial example:



Fellows' Planar Emulator Conjecture

Conjecture 8 (Fellows, 1989 – unpublished manuscript)

A connected graph has a finite planar emulator

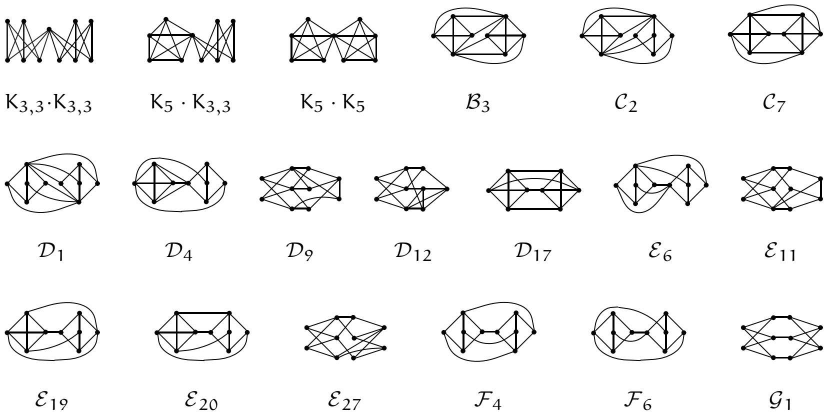
$\iff$ *it has a finite planar cover.*

Comparing Negami and Fellows

- Every planar cover is a planar emulator, too.
- Conv., some of the "no-planar-cover" args. extend to emulators:
  - the previous proof for $K_{3,5}$, via a clever trick, and
  - a proof for the first 19 of the proj. obstructions, too.
- So far, showing no example of an emulator that would not lead to a double planar cover (the only possibility by Negami's conjecture).
- How could one, actually, gain anything by using an emulator with duplicate neighbour? More edges take us only "away from planarity"!

Sample proof: two disjoint k-graphs

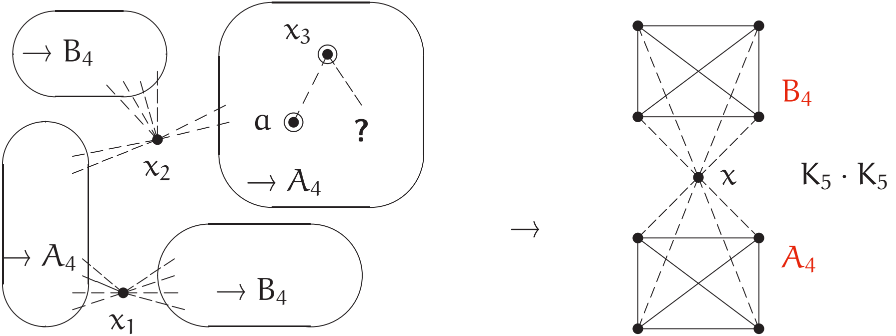
The first 19 of the connected projective obstructions fall into the same category—of two non-outerplanar pieces well-connected together…



All of these graphs present the same deep obstruction to planar emulability, as we will show next.

Theorem 9 (Negami / Archdeacon, 1988)
Neither of the graphs $K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$ *have a finite planar cover.*

Proof sketch. We choose the $K_5 \cdot K_5$ case for an illustration...



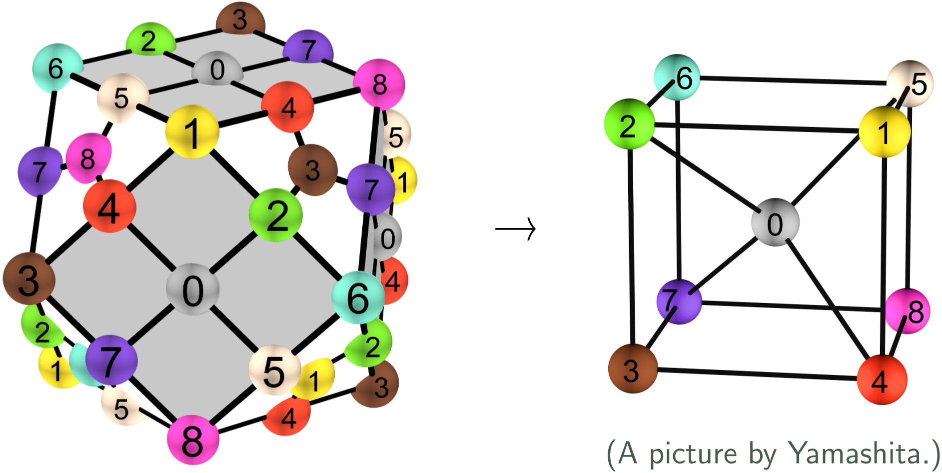
Each of A_4, B_4 is *non-outerplanar* by itself, and so the "pieces" of the assumed emulator mapping to A_4 and to B_4 are not outerplanar, too.

The latter is a big problem for connections to x... □

4 Surprising Fall of Fellows' Conjecture

Fact. The graph $K_{4,5} - 4K_2$ has no finite planar cover.

Theorem 10 (Rieck and Yamashita, 2008)
The graphs $K_{1,2,2,2}$ and $K_{4,5} - 4K_2$ do have finite planar emulators!!!



(A picture by Yamashita.)

Constructing more counterexamples

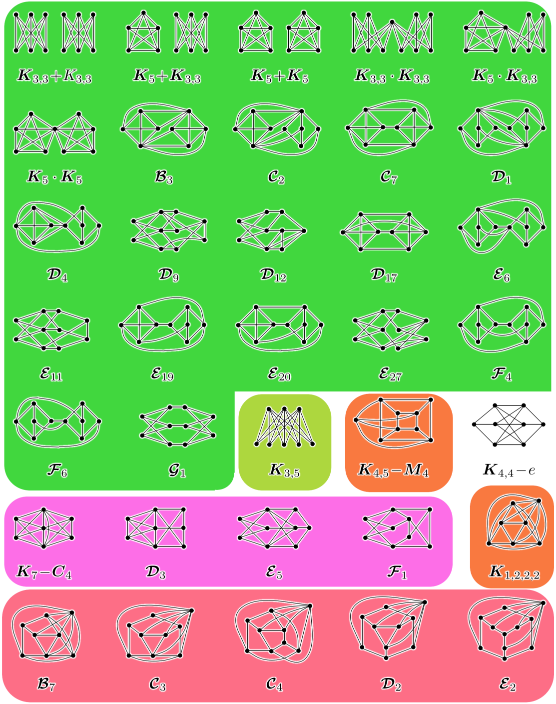
Theorem 11 (Chimani, Derka, Klusáček, and PH, 2011)
The graphs $\mathcal{B}_7$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{D}_2$, $\mathcal{E}_2$, and $\mathrm{K}_7 - \mathrm{C}_4$, $\mathcal{D}_3$, $\mathcal{E}_5$, $\mathcal{F}_1$ do have finite planar emulators, too.

Consequently, there remains only one out of the 32 connected projective obstructions for which planar emulability is not decided yet.

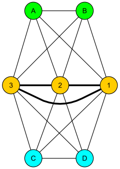
- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.
- So, let us study this class...

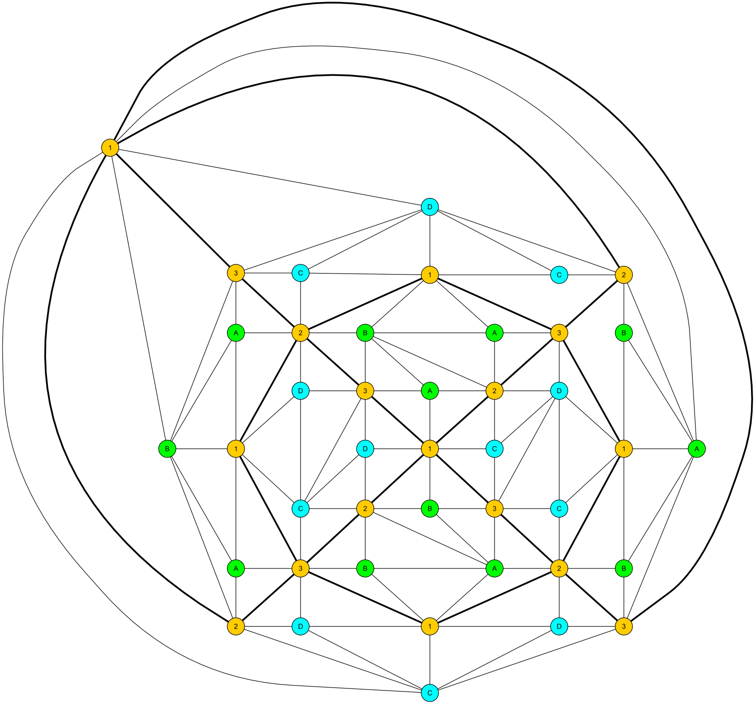
  In particular, how big is this difference?

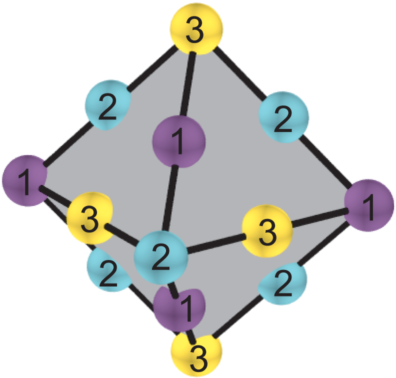
Graphically



K7-C4







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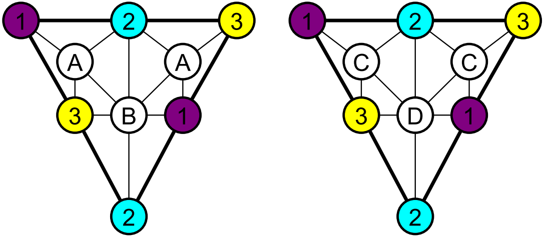
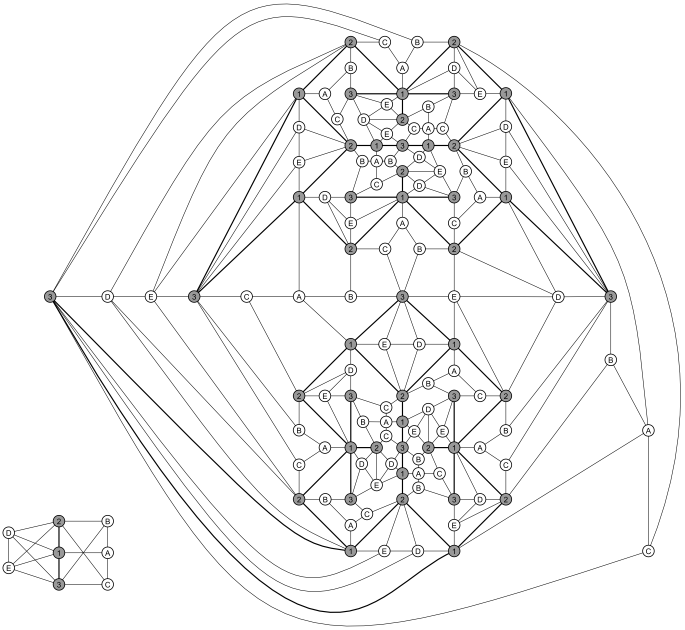
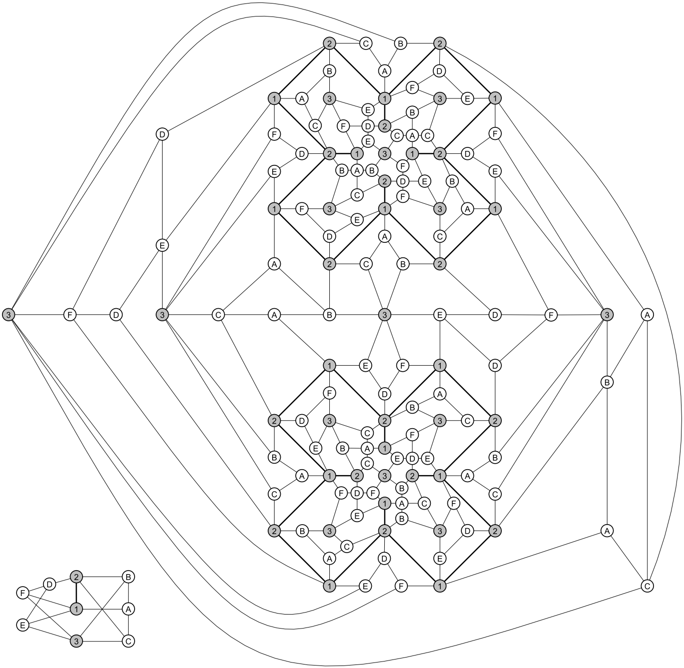
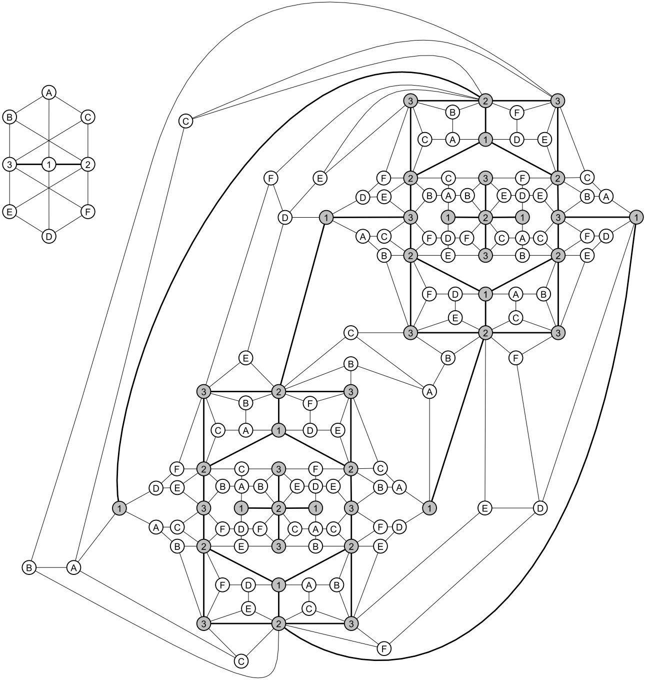


1 2 3
A A
3 B 1
2
1 2 3
C C
3 D 1
2



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Time for a Replacement Conjecture?

Conjecture 12 (Derka and PH)
There is a finite set $\mathcal{F}$ of graphs such that the following holds:

If a connected graph G has a finite planar emulator but no projective embedding, then G is a planar expansion of one of the members in $\mathcal{F}$.

Remarks:

- This is dir. inspired by Theorem 6 (possible counterex. to Negami). We know that $\mathcal{F}$ must be nonempty, though!
- With suitable high-level tools of structural graph theory, namely a

  splitter theorem for internally 4-connected graphs,

  this is just a finite computer search. . .
- Although, the search turned out very long and complex, and the available "splitter theorem" failed at some points.

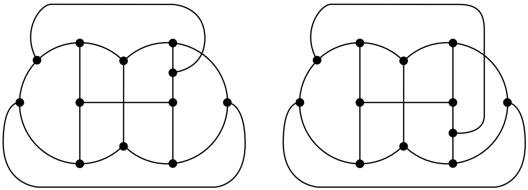
5 The cubic case

- Although we do not much understand the whole class of non-projective planar-emulable graphs

— is it essentially finite or infinite ? —

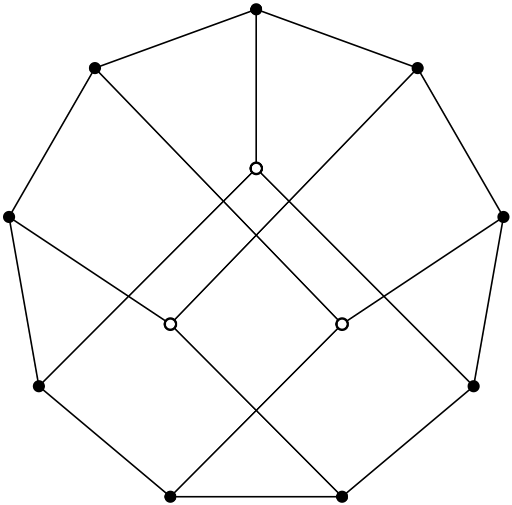
- it appears significant that no such cubic graph has been found.
- We can thus use this easier ground to perhaps train our techniques before attacking the full problem...

Theorem 13 (Derka and PH, 2013)
If a cubic nonprojective graph H *has a finite planar emulator, then* H *is a planar expansion of one of the following two graphs:*

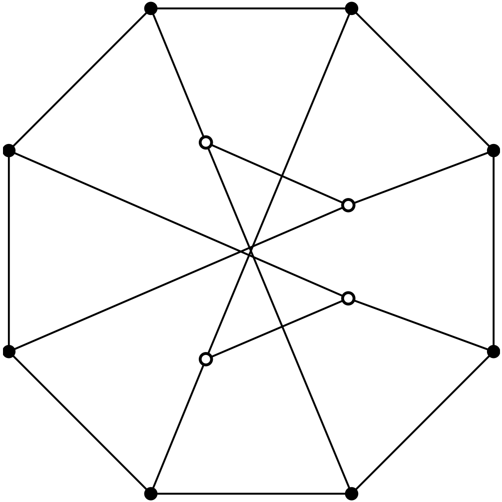


What about the remaining two graphs?

Trying a better picture...



And the second one. . .



Conclusion

- We do not seem to know enough about the planar emulability property to proceed with solving our conjecture. . .
- Besides the lack of a suitable *splitter theorem*,

  we mainly miss good methods to prove that a graph is

  not planar-emulable.
- The previous two graphs seem to be a good training ground for that
  - they do not seem to be planar-emulable, and
  - there are quite short proofs that they are not planar-coverable.
- At last, one should also look at the question whether the graph $K_{4,4} - e$ has a finite planar emulator or not.