Planar Emulators Conjecture Is Nearly True for Cubic Graphs

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* Joint work with Martin Derka, Brno / Waterloo.

Small math miracle for start. . .

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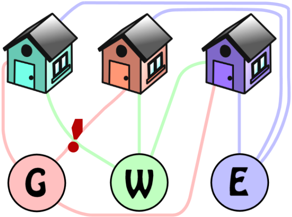
– turning nonplanar into planar



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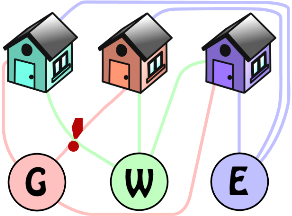

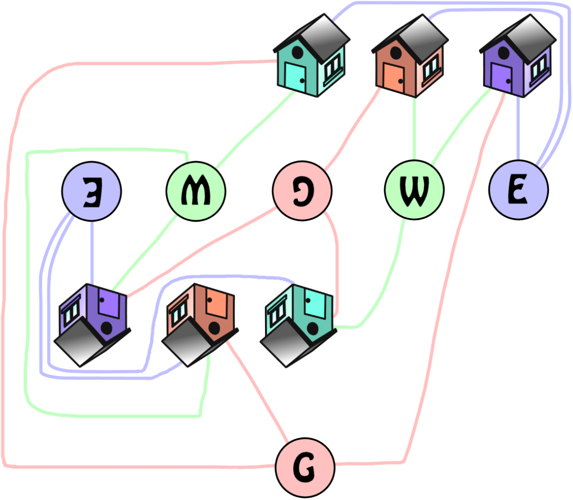

1 Definitions

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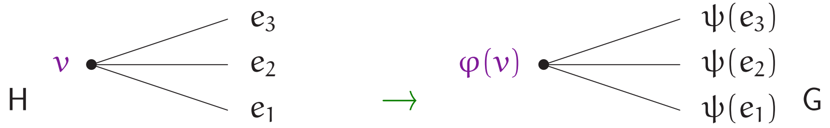
A graph H is a *cover* of a graph G if there exists a pair of onto mappings

(a *projection*) $\varphi : V(H) \to V(G), \quad \psi : E(H) \to E(G)$

such that ψ maps the edges incident with each vertex v in H bijectively onto the edges incident with $\varphi(v)$ in G.



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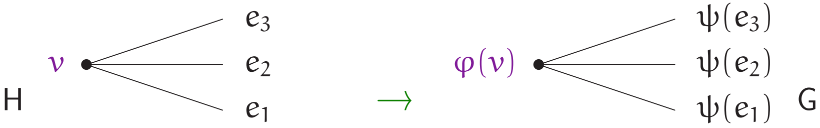


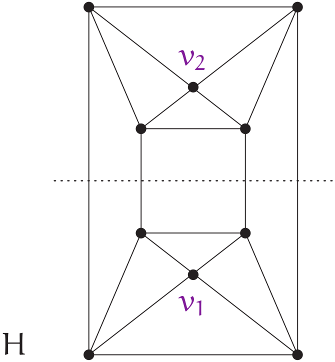
Remark. The edge $\psi(uv)$ has always ends $\varphi(u), \varphi(v)$, and hence only

$\varphi : V(H) \to V(G)$, the *vertex projection*,

is enough to be specified for simple graphs.

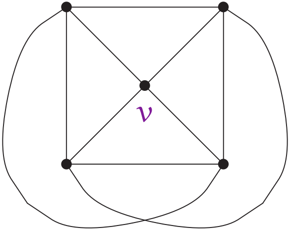
Planar covers

- We speak about a *planar cover* if H is a finite planar graph.



$\longrightarrow$

$\varphi(v_1) = \varphi(v_2) = v$



$G = K_5$

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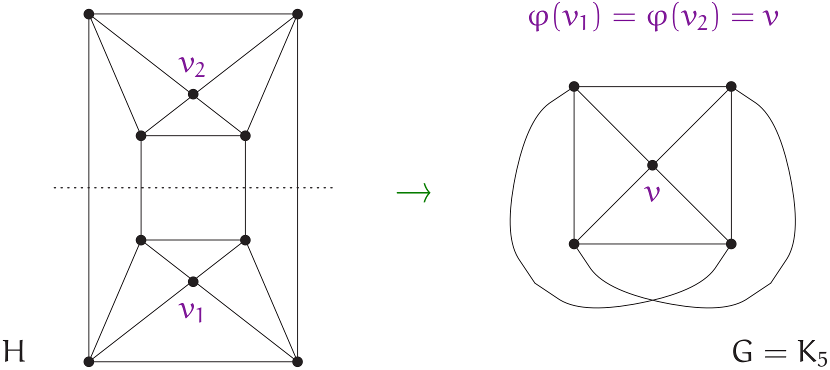

- Graph embedded in the *projective plane* has a double *planar cover*, *via the universal covering map from the sphere onto the proj. plane.*

Planar emulators

- $\varphi : \mathbf{V}(\mathbf{H}) \rightarrow \mathbf{V}(\mathbf{G})$, an *emulator* vs. a cover:

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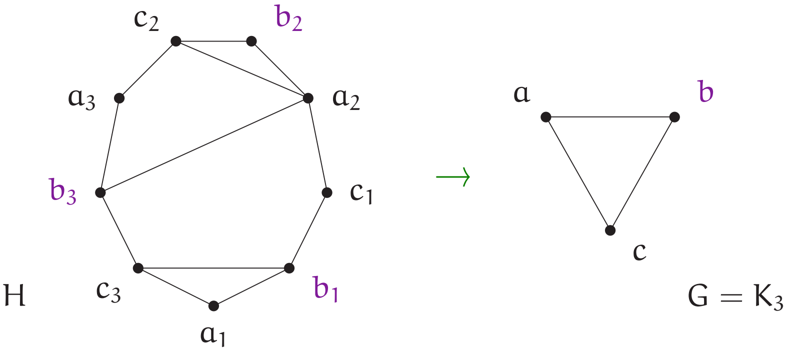

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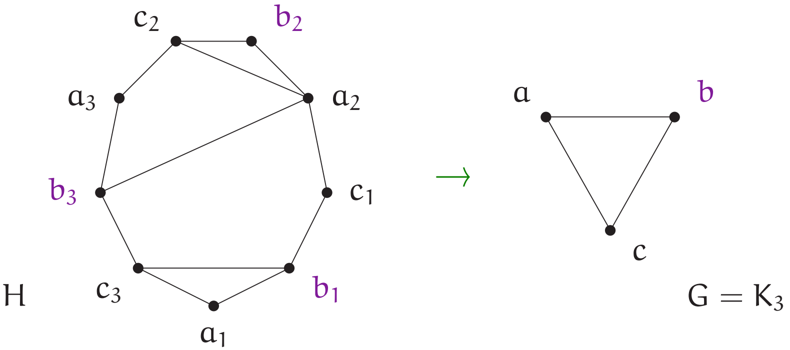

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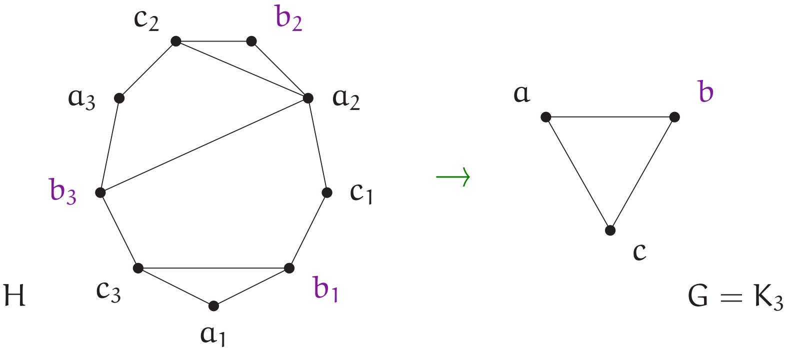

- Could a planar emulator be "more than" a planar cover?
- Not really, at least until 2008...

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Slow progress, and a sudden surprise

Long-term development around Negami's conjecture led to...

Theorem 3 (A+N+F+H, since 1998)
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The graphs $K_{1,2,2,2}$ and $K_{4,5}-4K_2$ do have finite planar emulators!!!

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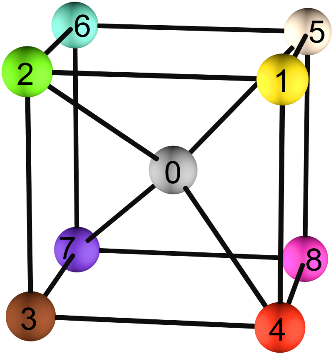
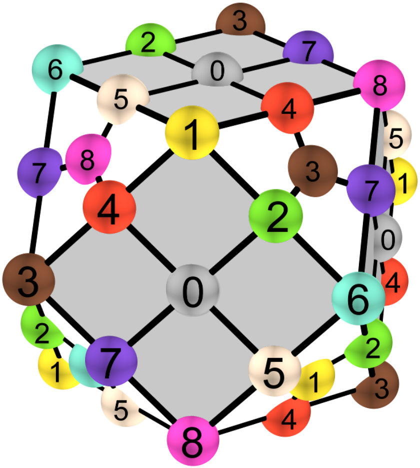
Theorem 4 (Rieck and Yamashita 2008)
The graphs $K_{1,2,2,2}$ *and* $K_{4,5}-4K_2$ *do have finite planar emulators!!!*

- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.
- So, let us study this class...

$K_{4,5} - 4K_2$



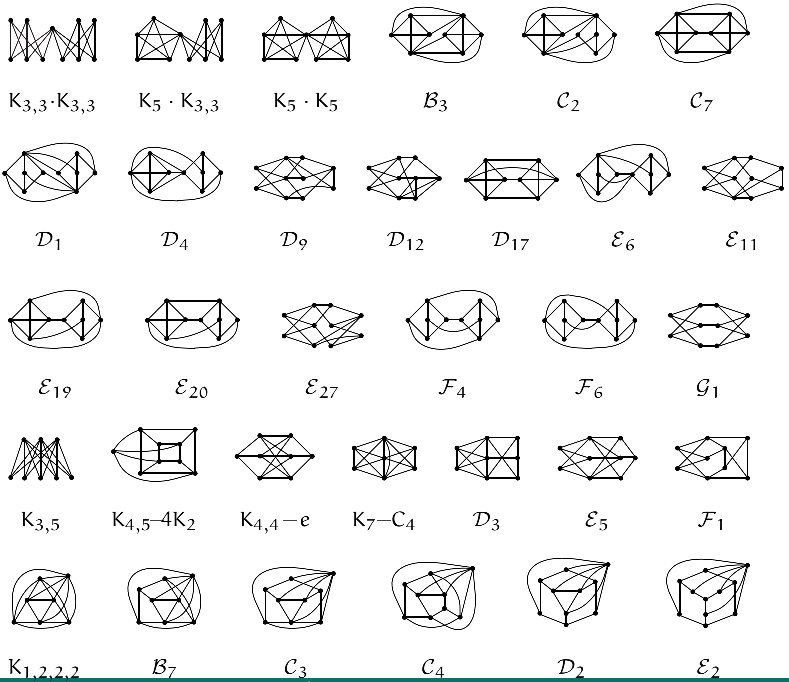
$\longleftarrow$



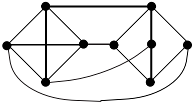
(A picture by Yamashita.)

3 What graphs do have planar emulators?

Recall...



Among the projective list



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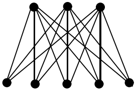

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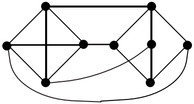

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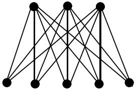

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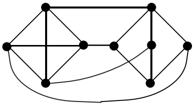

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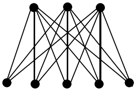

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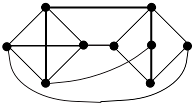

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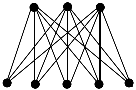

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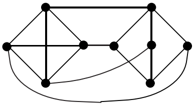

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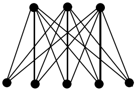

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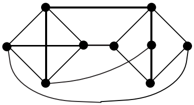

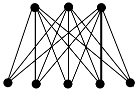

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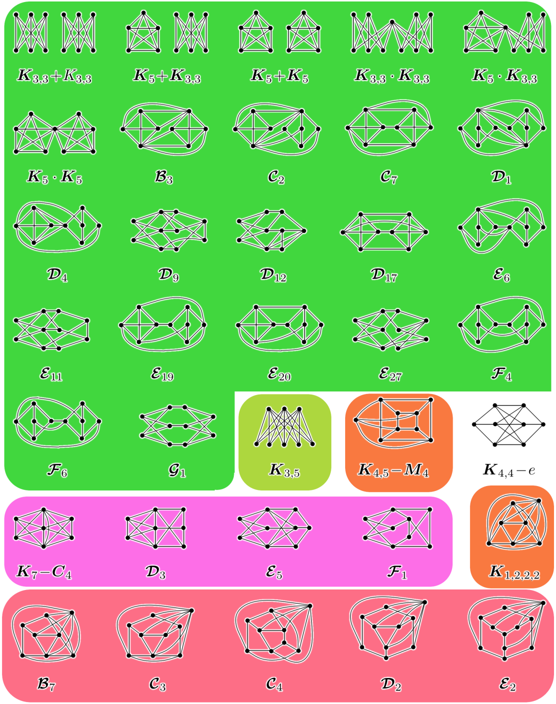
- and $K_7 - C_4$ and its whole family by [Klusáček, 2011].

Graphically



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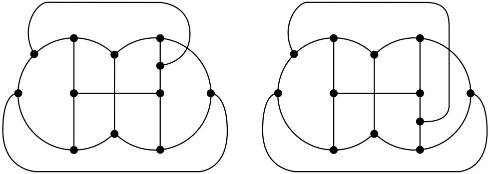
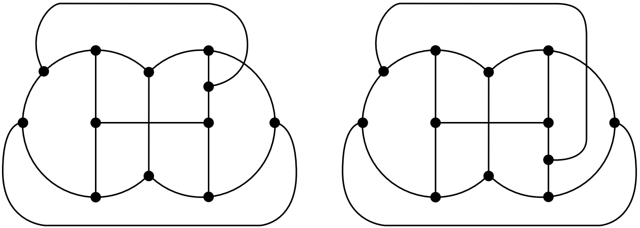
- it appears significant that no such cubic graph has been found.
- We can thus use this easier ground to perhaps train our techniques before attacking the full problem. . .

Theorem. If a cubic nonprojective graph H has a finite planar emulator, then H is a *planar expansion* of one of the following two graphs:



Proof:

G_1



G_2

- The two graphs – [Glover and Huneke, 1975]
  – two out of all *six cubic obstructions* for projective embeddability; while the other four have *two disjoint* $\Bbbk$*-graphs*.

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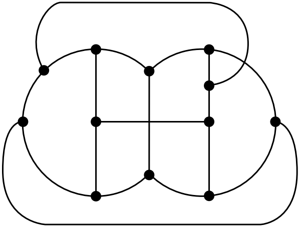

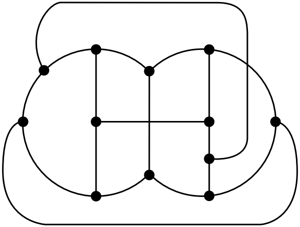

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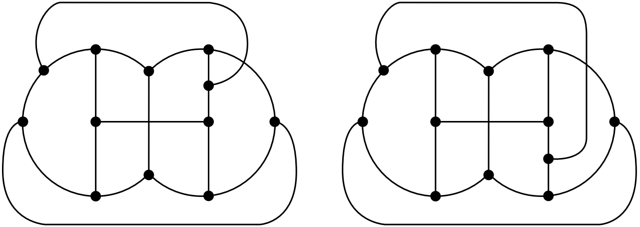

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- Cubic H $\Rightarrow$ $H = G_i +$ *bridges*
  where all the *bridge legs* subdivide the edges of G_i.

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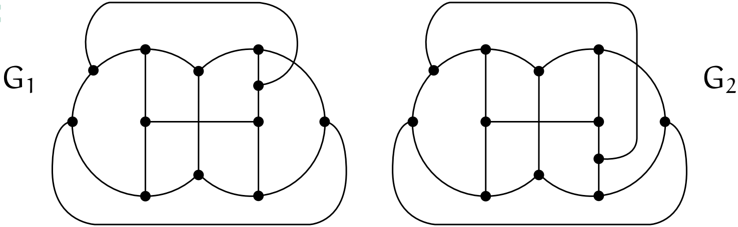

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- Cubic $H \Rightarrow$ $\boxed{H = G_i + \textit{bridges}}$
  where all the *bridge legs* subdivide the edges of G_i.
- Even a single bridge having legs on non-incident edges of G_i
  $\Rightarrow$ two disjoint k-graphs or a $K_{3,5}$ minor. [computer] OK

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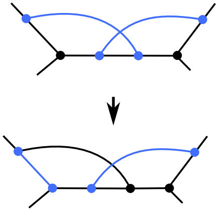

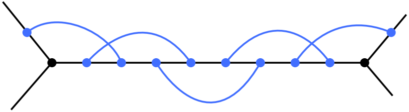

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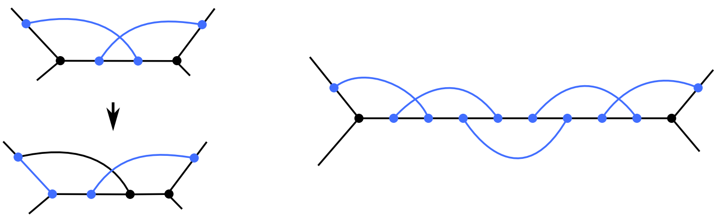

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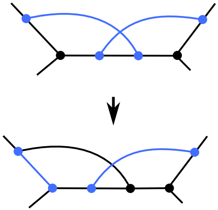

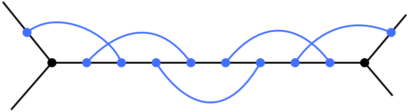

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- Hence the assign. of bridges to vert./edges of G_i is rigorous. OK

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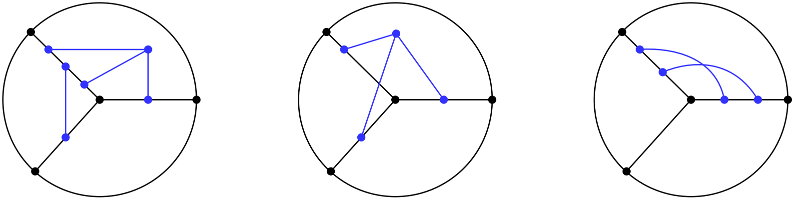

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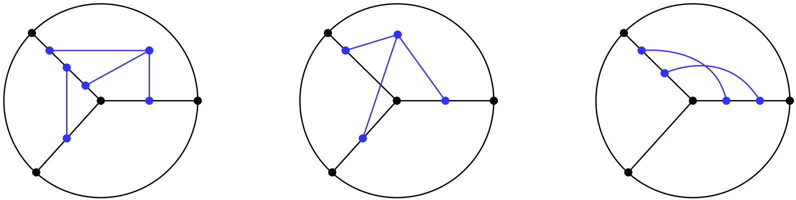

either *planar expansion* (left), or $K_{2,3}$ in the fragment (twice right),

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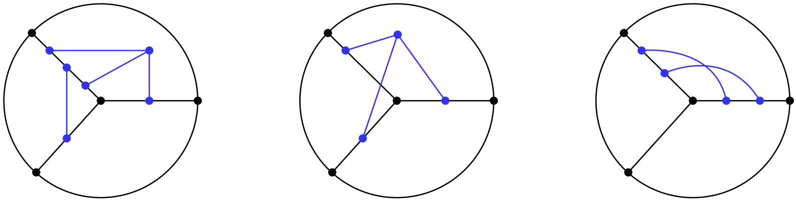

either *planar expansion* (left), or $K_{2,3}$ in the fragment (twice right),

$\Rightarrow$ two disjoint k-graphs. OK

- End of proof. □

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  - Negami $\iff$ true for $\mathcal{G} =$projective (and so false with emul.),
  - other classes, e.g., $\mathcal{G} =$ other nonorientable surface?