Planar Emulators Conjecture Is Nearly True for Cubic Graphs

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* Joint work with Martin Derka, FI MU Brno.

Small math miracle for start. . .

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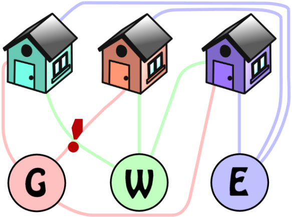
– turning nonplanar into planar



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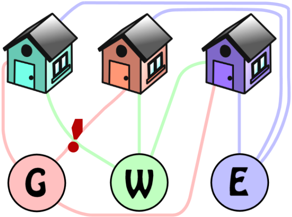

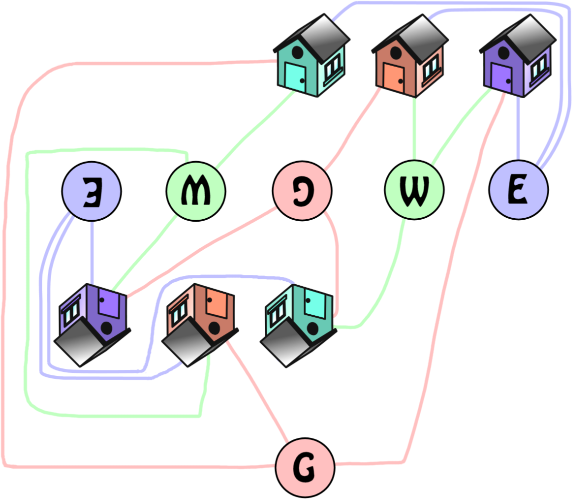

1 Definitions

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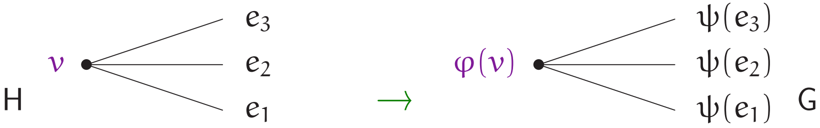
A graph H is a *cover* of a graph G if there exists a pair of onto mappings

(a *projection*) $\varphi : V(H) \to V(G), \quad \psi : E(H) \to E(G)$

such that ψ maps the edges incident with each vertex v in H bijectively onto the edges incident with $\varphi(v)$ in G.



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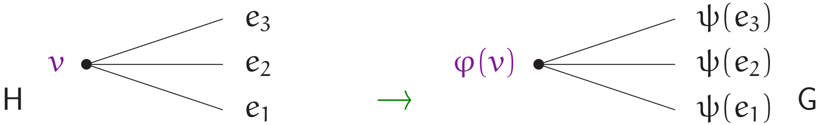


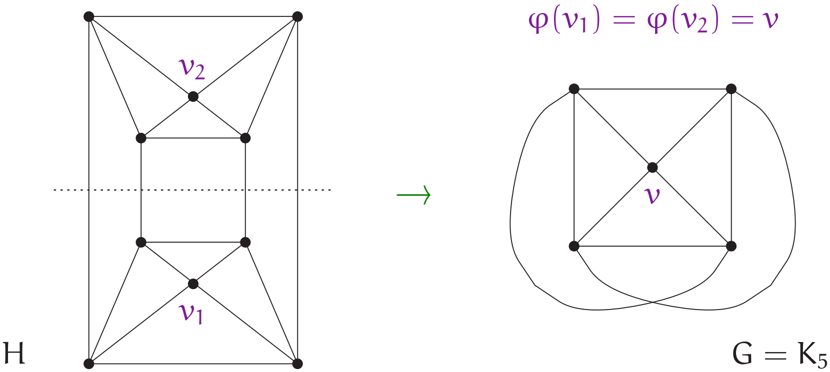
Remark. The edge $\psi(uv)$ has always ends $\varphi(u), \varphi(v)$, and hence only

$\varphi : V(H) \to V(G)$, the *vertex projection*,

is enough to be specified for simple graphs.

Planar covers

- We speak about a *planar cover* if H is a finite planar graph.



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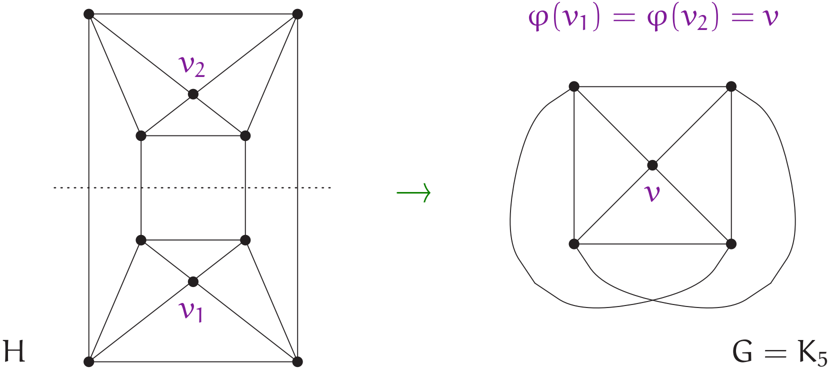

- Graph embedded in the *projective plane* has a double *planar cover*,

  via the universal covering map from the sphere onto the proj. plane.

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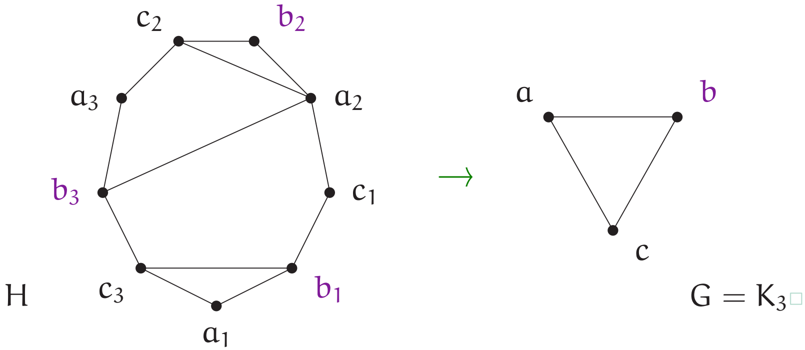

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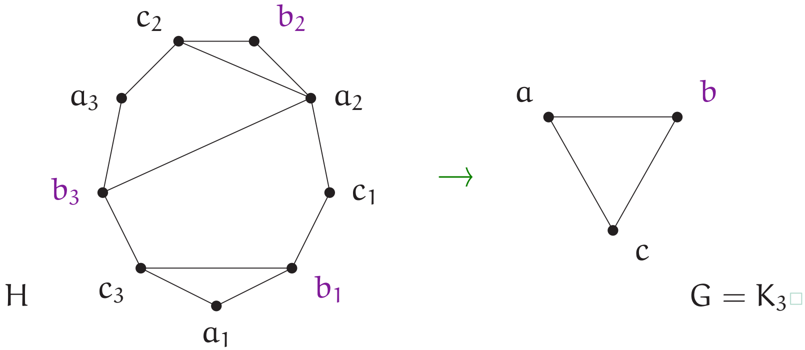

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- Not really, at least until 2008. . .

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Conjecture 1 (Negami, 1988)

A connected graph has a finite planar cover

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Theorem 4 (Rieck and Yamashita 2008)
The graphs $K_{1,2,2,2}$ *and* $K_{4,5} - 4K_2$ *do have finite planar emulators!!!*

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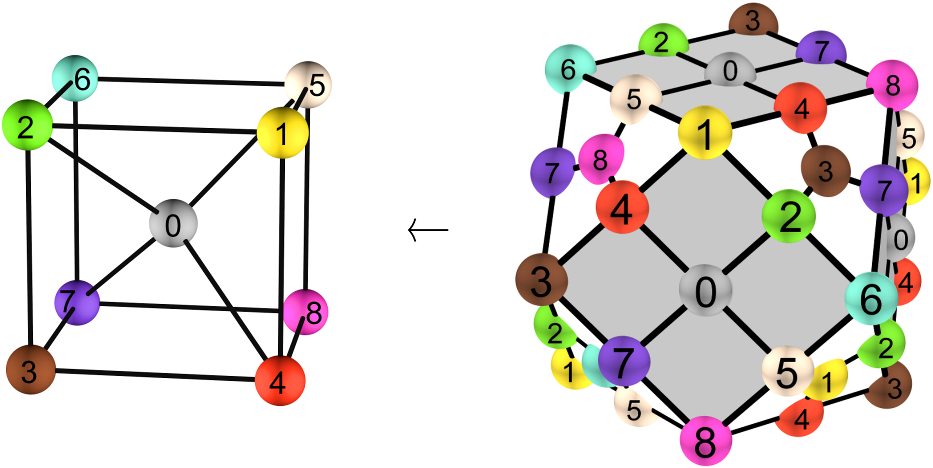
- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.
- So, let us study this class. . .

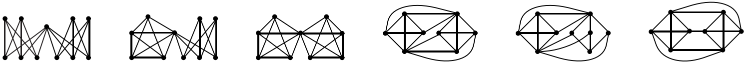
$K_{4,5} - 4K_2$



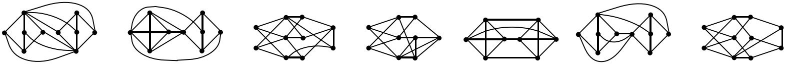
(A picture by Yamashita.)

3 What graphs do have planar emulators?

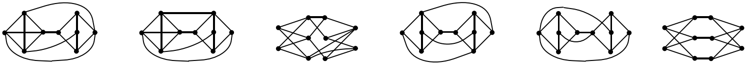
Recall. . .



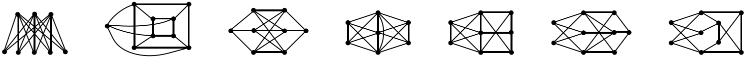
$K_{3,3} \cdot K_{3,3}$ $K_5 \cdot K_{3,3}$ $K_5 \cdot K_5$ $\mathcal{B}_3$ $\mathcal{C}_2$ $\mathcal{C}_7$



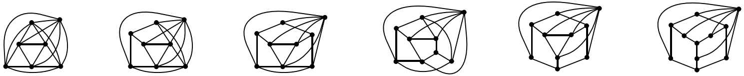
$\mathcal{D}_1$ $\mathcal{D}_4$ $\mathcal{D}_9$ $\mathcal{D}_{12}$ $\mathcal{D}_{17}$ $\mathcal{E}_6$ $\mathcal{E}_{11}$



$\mathcal{E}_{19}$ $\mathcal{E}_{20}$ $\mathcal{E}_{27}$ $\mathcal{F}_4$ $\mathcal{F}_6$ $\mathcal{G}_1$

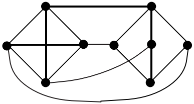


$K_{3,5}$ $K_{4,5}-4K_2$ $K_{4,4}-e$ K_7-C_4 $\mathcal{D}_3$ $\mathcal{E}_5$ $\mathcal{F}_1$

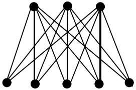


$K_{1,2,2,2}$ $\mathcal{B}_7$ $\mathcal{C}_3$ $\mathcal{C}_4$ $\mathcal{D}_2$ $\mathcal{E}_2$

Among the projective list



$\mathcal{E}_{20}$



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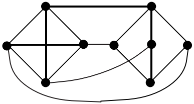

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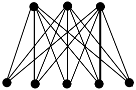

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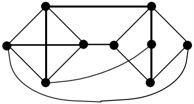

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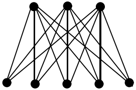

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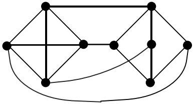

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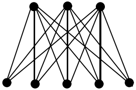

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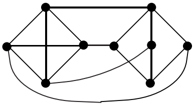

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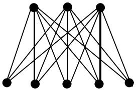

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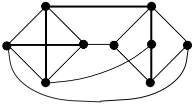

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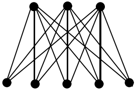

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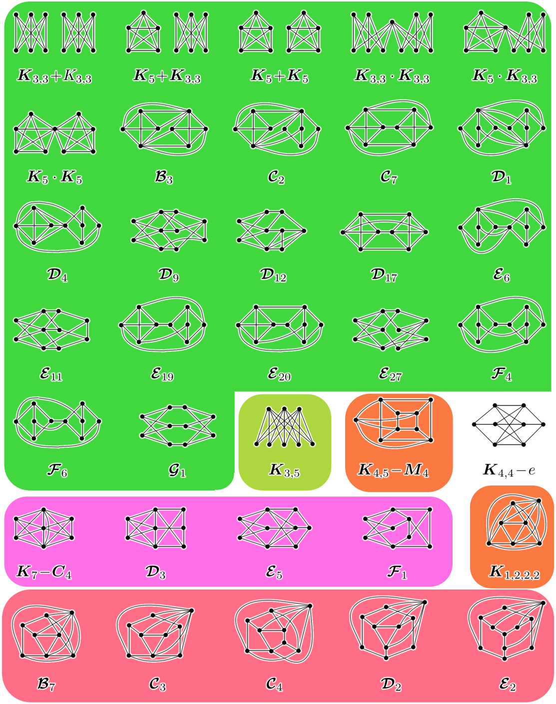
- and K_7-C_4 and its whole family by [Klusáček, 2011].

Graphically



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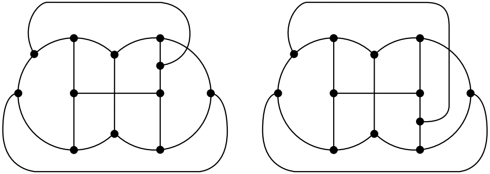
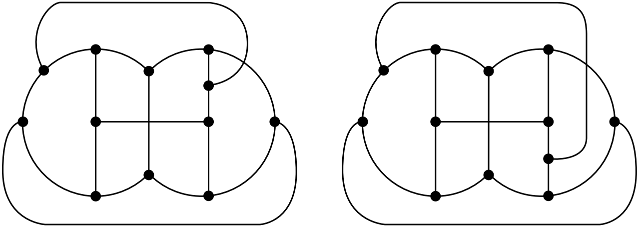
- it appears significant that no such cubic graph has been found.
- We can thus use this easier ground to perhaps train our techniques before attacking the full problem...

Theorem. If a cubic nonprojective graph H has a finite planar emulator, then H is a *planar expansion* of one of the following two graphs:



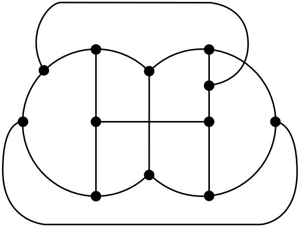
Proof:

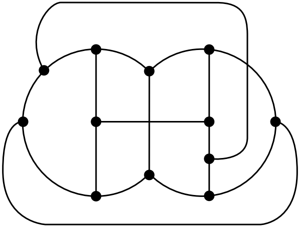


- The two graphs – [Glover and Huneke, 1975]
  – two out of all *six cubic obstructions* for projective embeddability; while the other four have *two disjoint* $\Bbbk$*-graphs*.

Proof:

G_1





G_2

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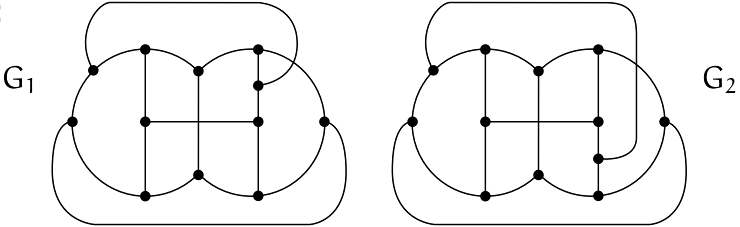

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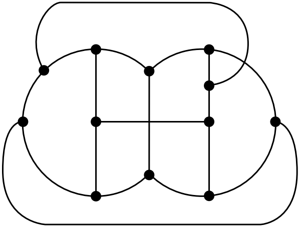
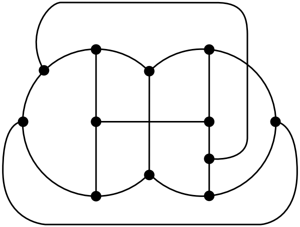
G$_1$   G$_2$

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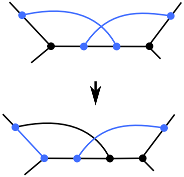
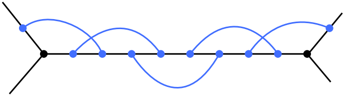
- Cubic H $\Rightarrow$ $\boxed{H = G_i + \textit{bridges}}$
  where all the *bridge legs* subdivide the edges of G_i.

- Even a single bridge having legs on non-incident edges of G_i
  $\Rightarrow$ two disjoint k-graphs or a $K_{3,5}$ minor. OK

All bridges trivial

- Must handle overlapping of bridges:





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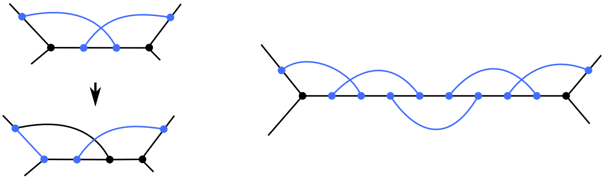

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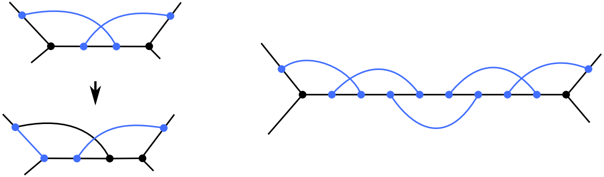

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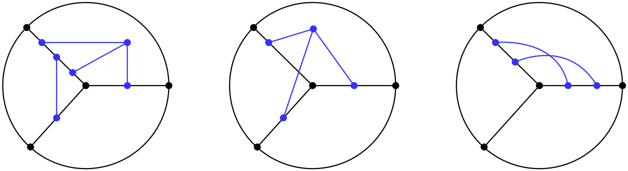

either *planar expansion* (left), or $K_{2,3}$ in the fragment (twice right).

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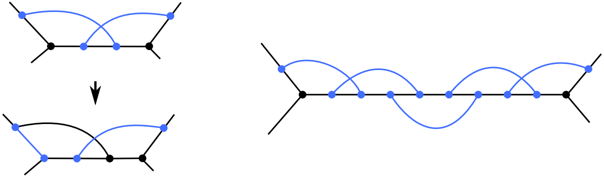

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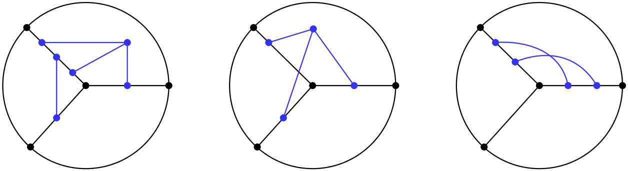

either *planar expansion* (left), or $K_{2,3}$ in the fragment (twice right).
$\Rightarrow$ two disjoint k-graphs. OK

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  - Holds true for $\mathcal{G} =$ outerplanar (so, k-outerplanar?),
  - Negami $\iff$ true for $\mathcal{G} =$projective (and so false with emul.),
  - other classes, e.g., $\mathcal{G} =$ other nonorientable surface?