# New Development in Planar Emulators

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* Joint work with Markus Chimani, Univ. Jena,
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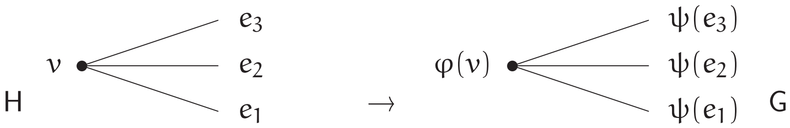
# 1 Definitions

*Motivation: Exploring the two graphs locally, we cannot see any difference...*

A graph $H$ is a *cover* of a graph $G$ if there exists a pair of onto mappings

(a *projection*) $\varphi : V(H) \to V(G), \quad \psi : E(H) \to E(G)$

such that $\psi$ maps the edges incident with each vertex $v$ in $H$
bijectively onto the edges incident with $\varphi(v)$ in $G$.



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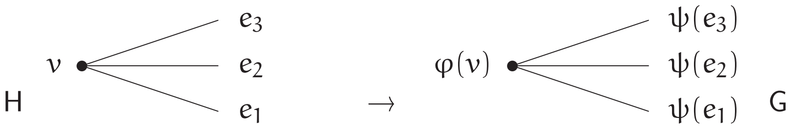


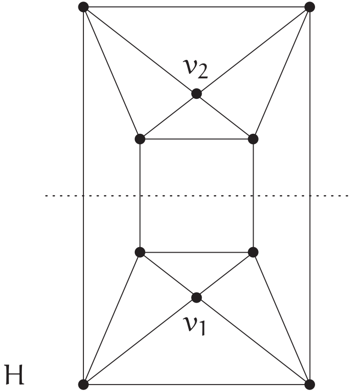

**Remark.** The edge $\psi(uv)$ has always ends $\varphi(u), \varphi(v)$, and hence only

$\varphi : V(H) \to V(G)$, the *vertex projection*,

is enough to be specified for simple graphs.

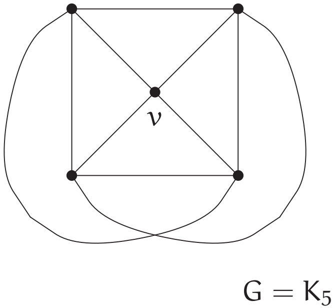
## Planar covers

- We speak about a *planar cover* if $H$ is a finite planar graph.




$\rightarrow$

$\varphi(v_1) = \varphi(v_2) = v$



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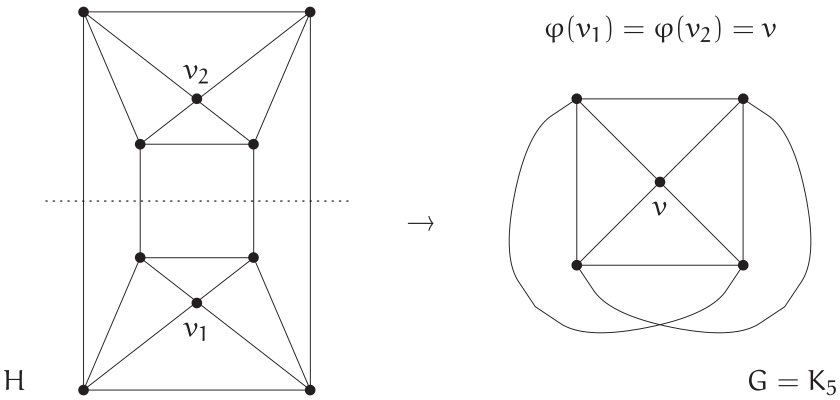
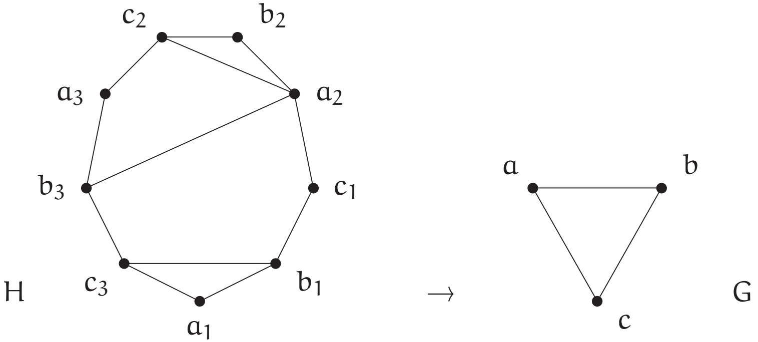



- Graph embedded in the *projective plane* has a double planar cover, via the universal covering map from the sphere onto the projective plane.

# Planar emulators

- $\varphi : V(H) \to V(G)$, an *emulator* vs. a cover:

...map the edges inc. with $v$ in H surjectively
onto the edges inc. with $\varphi(v)$ in G.

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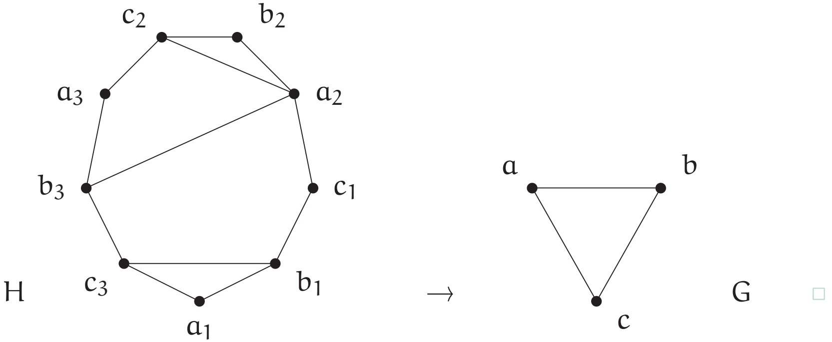


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- Not many remarkable results until 2008... Interesting at all?

# 2 Fellows' planar emulator conjecture

**Fact.** A planar cover is also a planar emulator.

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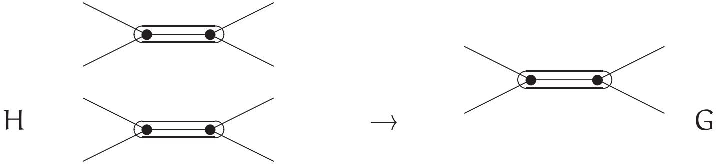
And via Negami's planar cover conjecture...

**Conjecture 2** (Kitakubo, 1991) *A connected graph has a finite planar emulator if and only if it embeds in the projective plane.*

# 3 Some useful properties

- If G has a planar cover, then so does every minor of G.



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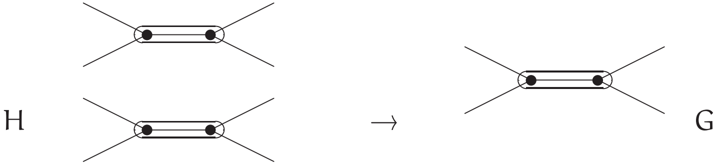
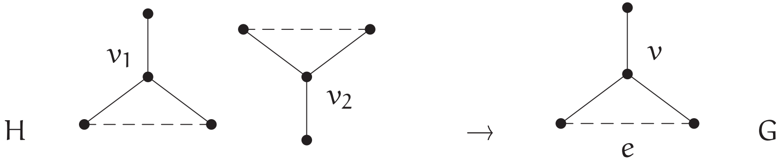



Consider $e$ between two neighbours of a cubic vertex.
If $G - e$ has a planar cover, then so does G.



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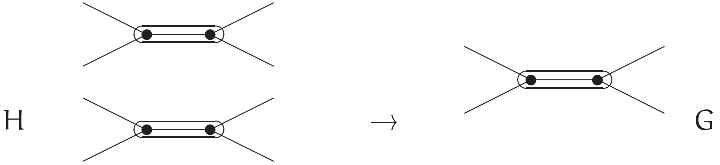


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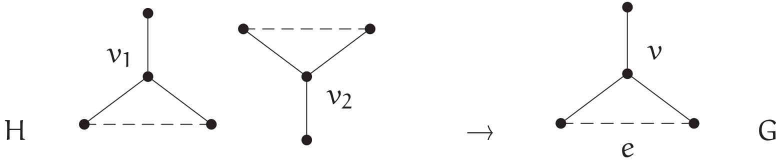


- Therefore, if $G$ has a planar cover, and $G'$ is obtained from $G$ by *YΔ-transformations*, then $G'$ has a planar cover, too.

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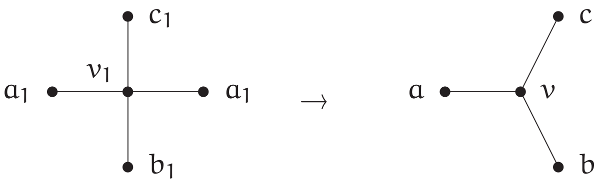

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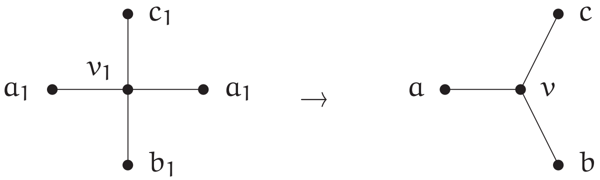


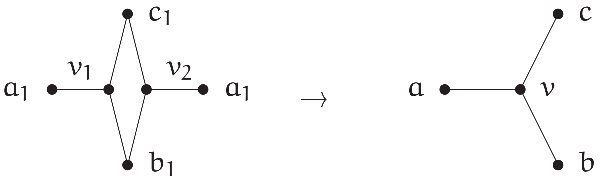

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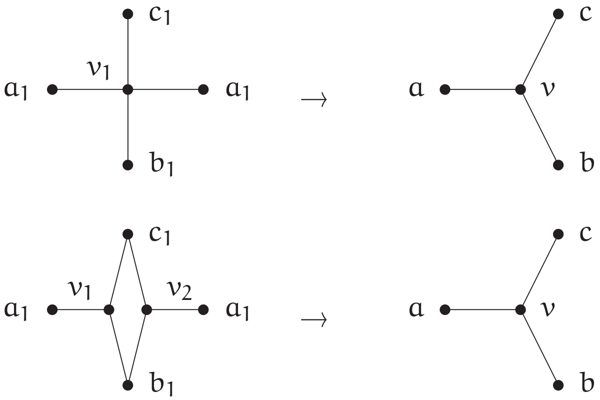


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*A connected graph has a finite planar cover / emulator if and only if it embeds in the projective plane.*

We recall the above basic properties...

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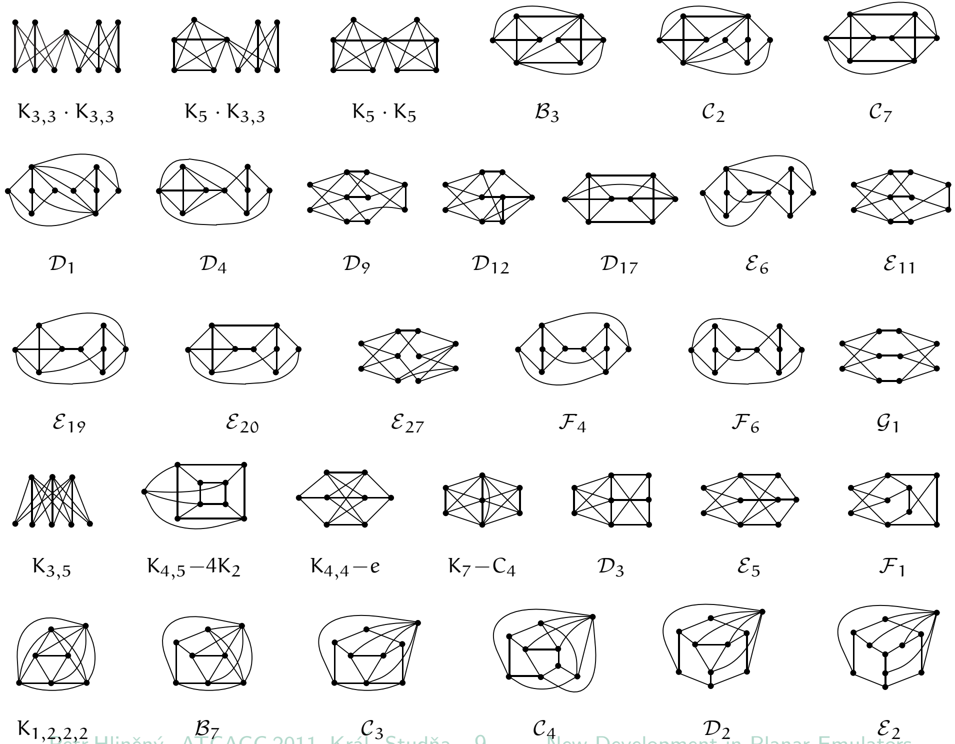
  Then G contains some F of the *forbidden minors* for the projective plane.
  We just have to show that this F has no finite planar cover / emulator.
- Furthermore, it is enough to consider only those F which are YΔ-transforms of some forbidden minor in G.



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$\mathcal{B}_3$
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**... and then...** Suddenly, Fellows' conjecture falls down...

**Fact.** The graph $K_{4,5} - 4K_2$ has no finite planar cover.

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- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.
- So, let us study this class. . . !

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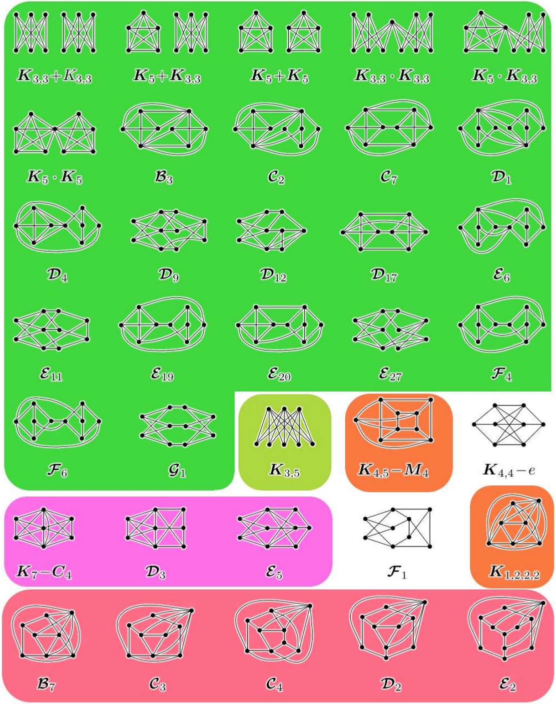
- and NEW $K_7 - C_4$ and (most of ??) its family! by [Klusáček, 2011].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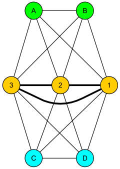
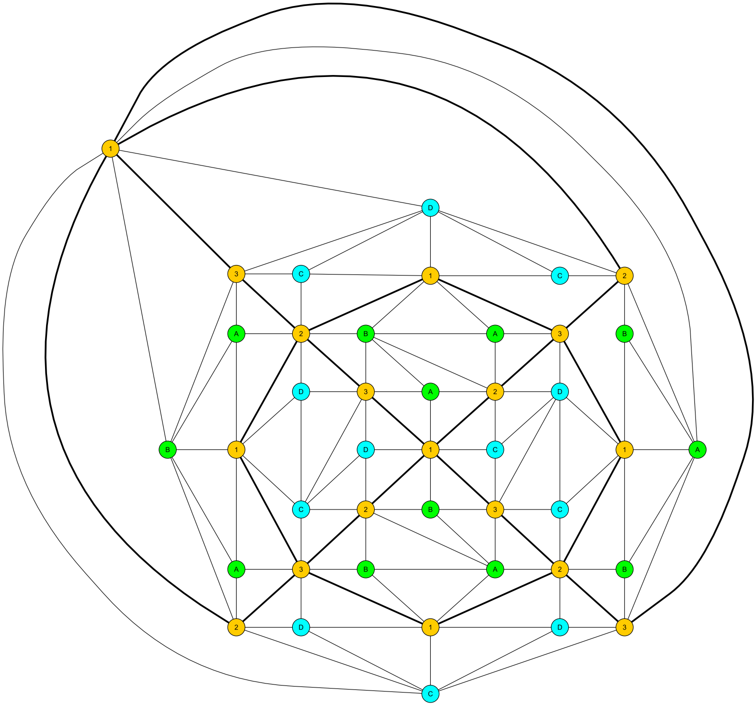
# K7-C4



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- And then study the strange class of those grahs having finite planar emulators!
  - Though the class originally looked quite similar to the projective-planar graphs, now ("after Klusáček") all has changed.
  - Any idea for a *new hypothesis*?
  - Any idea for a general structural result saying that the class of graphs having no minor in the "green picture" and possesssing certain connectivity (*internally 4-connected* enough? / maybe even $(5,3)$*-connectivity* would work?) is finite?