On Matroid Properties Definable in the MSO Logic

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Abstract. It has been proved by the author that all matroid properties definable in the monadic second-order (MSO) logic can be recognized in polynomial time for matroids of bounded branch-width which are represented by matrices over finite fields. (This result extends so called "MS_2-theorem" of graphs by Courcelle and others.) In this work we review the MSO theory of finite matroids and show some interesting matroid properties which are MSO-definable. In particular, all minor-closed properties are recognizable in such way.

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1 Introduction

The theory of parametrized complexity provides a background for analysis of difficult algorithmic problems which is finer than classical complexity theory. We postpone formal definitions till Section 3. Briefly saying, a problem is called "fixed-parameter tractable" if there is an algorithm having running time with the (possible) super-polynomial part separated in terms of some natural "parameter", which is supposed to be small even for large input in practice. (Successful practical applications of this concept are known, for example, in computational biology or in database theory.)

We are interested in algorithmic problems that are parametrized by a "tree-like" structure of the input objects. Graph "branch-width" is closely related to well-known tree-width [13], but a branch decomposition does not refer to vertices, and so branch-width directly generalizes from graphs to matroids. It follows from works of Courcelle [2] and Bodlaender [1] that all graph problems definable in the monadic second-order logic can be solved in linear time for graphs of bounded tree-width. Those include many notoriously hard problems like 3-colouring, Hamiltonicity, etc.

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We study and present analogous results for matroids representable over finite fields. The motivation of our research is mainly theoretical — to show how the mentioned complexity phenomenon extends from graphs to a much larger class of combinatorial objects, and to stimulate further research interest in matroid branch-width and the complexity of matroid problems. (Unfortunately, wide generality of our approach leads to impractically huge constants involved in the algorithms, such as in Theorem 4.1.) Since not all computer scientists are familiar with structural matroid theory or with parametrized complexity, we give a basic overview of necessary concepts in the next two sections.

2 Matroids and Branch-Width

We refer to Oxley [12] for matroid terminology. A *matroid* is a pair $M = (E, \mathcal{B})$ where $E = E(M)$ is the ground set of M (elements of M), and $\mathcal{B} \subseteq 2^E$ is a nonempty collection of *bases* of M. Moreover, matroid bases satisfy the "exchange axiom"; if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there is $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$. We consider only finite matroids. Subsets of bases are called *independent sets*, and the remaining sets are *dependent*. Minimal dependent sets are called *circuits*. All bases have the same cardinality called the *rank* $\mathrm{r}(M)$ of the matroid. The *rank function* $\mathrm{r}_M : 2^E \to \mathbb{N}$ of M tells the maximal cardinality $\mathrm{r}_M(X)$ of an independent subset of a set $X \subseteq E(M)$.

If G is a graph, then its *cycle matroid* on the ground set $E(G)$ is denoted by $M(G)$. The bases of $M(G)$ are the (maximal) spanning forests of G, and the circuits of $M(G)$ are the cycles of G. Another example of a matroid is a finite set of vectors with usual linear dependency. If $\boldsymbol{A}$ is a matrix, then the matroid formed by the column vectors of $\boldsymbol{A}$ is called the *vector matroid* of $\boldsymbol{A}$, and denoted by $M(\boldsymbol{A})$. The matrix $\boldsymbol{A}$ is a *representation* of a matroid $M \simeq M(\boldsymbol{A})$. We say that the matroid $M(\boldsymbol{A})$ is $\mathbb{F}$-*represented* if $\boldsymbol{A}$ is a matrix over a field $\mathbb{F}$.

The *dual* matroid M^* of M is defined on the same ground set E, and the bases of M^* are the set-complements of the bases of M. The dual rank function satisfies $\mathrm{r}_{M^*}(X) = |X| - \mathrm{r}(M) + \mathrm{r}_M(E - X)$. A set X is *coindependent* in M if it is independent in M^*. An element e of M is called a *loop* (a *coloop*), if $\{e\}$ is dependent in M (in M^*). The matroid $M \setminus e$ obtained by *deleting* a non-coloop element e is defined as $(E - \{e\}, \mathcal{B}^-)$ where $\mathcal{B}^- = \{B : B \in \mathcal{B}, e \notin B\}$. The matroid M/e obtained by *contracting* a non-loop element e is defined using duality $M/e = (M^* \setminus e)^*$. (This corresponds to contracting an edge in a graph.)

A *minor* of a matroid is obtained by a sequence of deletions and contractions of elements. Since these operations naturally commute, a minor M' of a matroid M can be uniquely expressed as $M' = M \setminus D/C$ where D are the coindependent deleted elements and C are the independent contracted elements. A matroid family $\mathcal{M}$ is *minor-closed* if $M \in \mathcal{M}$ implies that all minors of M are in $\mathcal{M}$. A matroid N is called an *excluded minor* (also known as "forbidden") for a minor-closed family $\mathcal{M}$ if $N \notin \mathcal{M}$ but $N' \in \mathcal{M}$ for all proper minors N' of N.

The connectivity function λ_M of a matroid M is defined for all subsets $A \subseteq E = E(M)$ by $\lambda_M(A) = \mathrm{r}_M(A) + \mathrm{r}_M(E - A) - \mathrm{r}(M) + 1$. Notice that $\lambda_M(A) =$

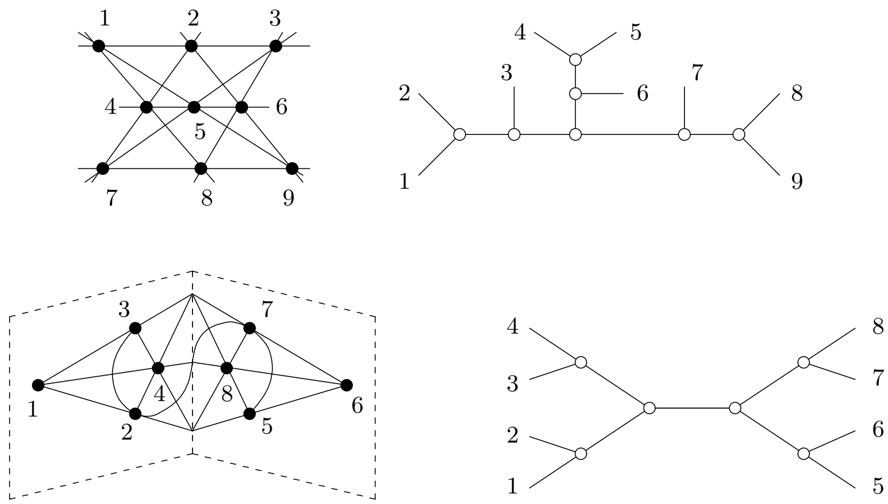

Fig. 1. Two examples of width-3 branch decompositions of the Pappus matroid (top left, rank 3) and of the binary affine cube (bottom left, rank 4). Here the lines depict linear dependencies between matroid elements.

$\lambda_M(E - A)$. Is is also routine to verify that $\lambda_M(A) = \lambda_{M^*}(A)$, i.e. matroid connectivity is dual-invariant. A subset $A \subseteq E$ is *k-separating* if $\lambda_M(A) \le k$. A partition $(A, E-A)$ is called a *k-separation* if A is k-separating and both $|A|, |E-A| \ge k$. For $n > 1$, the matroid M is called *n-connected* if it has no k-separation for $k = 1, 2, \ldots, n-1$, and $|E(M)| \ge 2n-2$. (A connected matroid corresponds to a vertex 2-connected graph. Geometric interpretation of a k-separation (A, B) is that the spans of A and of B intersect in a subspace of rank less than k.)

Let $\ell(T)$ denote the set of leaves of a tree T. A *branch decomposition* of a matroid M is a pair (T, τ) where T is a tree of maximal degree three, and τ is a bijection of $E(M)$ onto $\ell(T)$. Let f be an edge of T, and T_1, T_2 be the connected components of $T - f$. The *width* of an edge f in T is $\lambda_M(A) = \lambda_M(B)$, where $A = \tau^{-1}(\ell(T_1))$ and $B = \tau^{-1}(\ell(T_2))$. The width of the branch decomposition (T, τ) is maximum of the widths of all edges of T, and the *branch-width* of M is the minimal width over all branch decompositions of M. If T has no edge, then we take its width as 0.

An example of a branch decomposition is presented in Fig. 1. Notice that matroid branch-width is invariant under duality. It is straightforward to verify that branch-width does not increase when taking minors: Let (T, τ) be a branch decomposition of a matroid M. Say, up to duality, that $M' = M \setminus e$. We form T' from T by deleting the leaf $\tau(e)$, and set τ' to be τ restricted to $E(M')$. Then, for any partition (A, B) of $E(M)$ given by an edge f in T, we have obvious $\lambda_{M'}(A - \{e\}) \le \lambda_M(A)$, and so the width of (T', τ') is not bigger than the width of (T, τ) for M.

We remark that branch-width of a graph G is defined analogously, using the connectivity function λ_G where $\lambda_G(F)$ for $F \subseteq E(G)$ is the number of vertices incident both with F and $E(G) - F$. Clearly, branch-width of a graph G is never smaller than branch-width of its cycle matroid $M(G)$. It is still an open conjecture that these numbers are actually equal. On the other hand, branch-width is within a constant factor of tree-width in graphs [13].

Lastly in this section we mention few words about relations of matroid theory to computer science. As the reader surely knows, a greedy algorithm on a matroid is one of the basic tools in combinatorial optimization. That is why matroids naturally arise in a number of optimization problems; such as the minimum spanning tree or job assignment problems. More involved applications of matroids in combinatorial optimization could be found in numerous works of Edmonds, Cunningham and others. Besides that, the concept of branch-width has attracted increasing attention among matroid theorists recently, and several deep results of Robertson-Seymour's graph minor theory have been extended from graphs to matroids representable over finite fields; such as [6].

Robertson-Seymour's theory has been followed by many interesting algorithmic applications on graphs (mostly related to tree-width or branch-width). Therefore we think it is right time now to look at complexity aspects of branch-width in matroid problems. For example, we have given a straightforward polynomial algorithm for computation of the Tutte polynomial [10] on a representable matroid of bounded branch-width. (It seems that matroids present a more suitable model than graphs for computing the Tutte polynomial on structures of bounded tree-/branch-width.) As yet another motivation we remark that linear codes over a finite field $\mathbb{F}$ are in a direct correspondence with $\mathbb{F}$-represented matroids.

3 Parametrized Complexity

When speaking about *parametrized complexity*, we closely follow Downey and Fellows [4]. Here we present the basic definition of parametrized tractability. For simplicity, we restrict the definition to decision problems, although an extension to computation problems is straightforward. Let Σ be the input alphabet. A *parametrized problem* is an arbitrary subset $A^p \subseteq \Sigma^* \times \mathbb{N}$. For an instance $(x, k) \in A^p$, we call k the *parameter* and x the input for the problem. (The parameter is sometimes implicit in the context.)

We say that a parametrized problem A^p is *(nonuniformly) fixed-parameter tractable* if there is a sequence of algorithms $\{\mathcal{A}_i : i \in \mathbb{N}\}$, and a constant c; such that $(x, k) \in A^p$ iff the algorithm $\mathcal{A}_k$ accepts (x, k), and that the running time of $\mathcal{A}_k$ on (x, k) is $O(|x|^c)$ for each k. Similarly, a parametrized problem A^p is *uniformly fixed-parameter tractable* if there is an algorithm $\mathcal{A}$, a constant c, and an arbitrary function $f : \mathbb{N} \to \mathbb{N}$; such that $(x, k) \in A^p$ iff the algorithm $\mathcal{A}$ accepts (x, k), and that the running time of $\mathcal{A}$ on (x, k) is $O(f(k) \cdot |x|^c)$.

There is a natural correspondence of a parametrized problem A^p to an ordinary problem $A = \{\langle x, k\rangle : (x, k) \in A^p\}$ (for example, the problem of a

k-vertex cover in a graph), or to a problem $A' = \{x : \exists k\ (x,k) \in A^p\}$ if k is not "directly involved" in the question (such as a Hamiltonian cycle in a graph of tree-width k). On the other hand, an ordinary problem may have several natural parametrized versions respecting different parameters. We remark that the parameter is formally a natural number, but that may encode arbitrary finite structures in a standard way.

As we have already noted above, our interest is in parametrized problems where the parameter is branch-width (tree-width). Inspired by the algorithm of Bodlaender [1], we have shown that branch-width of matroids represented over finite fields is fixed parameter tractable, and that, moreover, we could efficiently construct a branch decomposition. Let $\mathcal{B}_t$ denote the class of all matroids of branch-width at most t. We have proved the following:

Theorem 3.1. (PH [9]) *Let $t \geq 1$ be fixed, and let $\mathbb{F}$ be a finite field. Suppose that $\boldsymbol{A}$ is an $r \times n$ matrix over $\mathbb{F}$ ($r \leq n$) such that the represented matroid $M(\boldsymbol{A}) \in \mathcal{B}_t$. Then there is an algorithm that finds a branch decomposition of the matroid $M(\boldsymbol{A})$ of width at most $3t$ in time $O(n^3)$.*

Actually, our algorithm directly constructs a so called "parse tree" for the mentioned branch decomposition.

Unfortunately, the algorithm in Theorem 3.1 does not necessarily produce the optimal branch decomposition. On the other hand, there are finitely many excluded minors for the class $\mathcal{B}_k$ for each k, and these excluded minors are constructed algorithmically since they have size at most $\frac{1}{5}(6^{k+1} - 1)$ by [5]. Hence, in this particular case, we can extend the idea in Theorem 5.2 to show:

Corollary 3.2. *Let $\mathbb{F}$ be a finite field. Suppose that $\boldsymbol{A}$ is a given matrix over $\mathbb{F}$. Then branch-width of the matroid $M(\boldsymbol{A})$ is uniformly fixed parameter tractable.*

4 MSO Logic of Matroids

The *monadic second-order (MSO) theory of matroids* uses language based on the monadic second-order logic. The syntax includes variables for matroid elements and element sets, the quantifiers $\forall, \exists$ applicable to these variables, the logical connectives $\wedge, \vee, \neg$, and the following predicates:

1. $=$, the equality for elements and their sets,
2. $e \in F$, where e is an element variable and F is an element set variable,
3. $\mathit{indep}(F)$, where F is an element set variable, and the predicate tells whether F is independent in the matroid.

Moreover, we write $\phi \rightarrow \psi$ to stand for $\neg\phi \vee \psi$, and $X \subseteq Y$ for $\forall x(x \in Y \vee x \notin X)$.

Notice that the "universe" of a formula (the model in logic terms) in the above theory is one particular matroid. To give a better feeling for the MSO theory of matroids, we provide few simple predicates now. We write $\mathit{basis}(B) \equiv \mathit{indep}(B) \wedge \forall e\big(e \in B \vee \neg\, \mathit{indep}(B \cup \{e\})\big)$ where $\mathit{indep}(B \cup \{e\})$ is a shortcut for obvious $\exists X\big(\, \mathit{indep}(X) \wedge e \in X \wedge B \subseteq X \wedge \forall x(x = e \vee x \in B \vee x \notin X)\big)$. Similarly, we write a predicate $circuit(C) \equiv \neg\, indep(C) \wedge \forall e\big(e \in C \rightarrow indep(C - \{e\})\big)$ where $indep(C-\{e\})$ is a shortcut for $\exists X\,\big(\, indep(X) \wedge e \notin X \wedge X \subseteq C \wedge \forall x(x = e \vee x \notin C \vee x \in X)\big)$.

Let us now look at the (graph) property of being Hamiltonian. In matroid language, that means to have a circuit containing a basis. So we may write a sentence $hamilton \equiv \exists C\big(\, circuit(C) \wedge \exists e\; basis(C - \{e\})\big)$. A related matroidal property is to be a *paving matroid* M — i.e., to have all circuits C in M of size $|C| \geq \mathrm{r}(M)$. Let us explain this sample property in detail. Since $C - \{e\}$ is independent for each $e \in C$ by definition of a circuit, we have $|C| \leq \mathrm{r}(M) + 1$ for any circuit C in M. Considering a basis $B \supseteq C - \{e\}$ and the inequality $|C| \geq \mathrm{r}(M) = |B|$ valid in a paving matroid, we conclude that there is an element f such that $B \subseteq C \cup \{f\}$. The converse also holds. Hence we express $paving \equiv \forall C\big[\, circuit(C) \rightarrow \exists f, B\big(B \subseteq C \cup \{f\} \wedge basis(B)\big)\big]$.

The reason why we are looking for properties definable in the MSO logic of matroids is, that such properties can be recognized in polynomial time for matroids of bounded branch-width over finite fields. The following result is based on a finite-state recognizability of matroidal MSO properties, proved by the author in [8], and on Theorem 3.1.

Theorem 4.1. (PH [7–9]) *Let $\mathbb{F}$ be a finite field. Assume that $\mathcal{M}$ is a class of matroids defined in one of the following ways;*
(a) there is an MSO sentence ϕ such that $M \in \mathcal{M}$ iff ϕ is true on M, or
(b) there is a sequence of MSO sentences $\{\phi_k : k = 1, 2, \ldots\}$ and, for all $k \geq 1$ and matroids $M \in \mathcal{B}_k$, we have $M \in \mathcal{M}$ iff ϕ_k is true on M.
Suppose that $\mathbf{A}$ is an n-column matrix over $\mathbb{F}$ such that $M(\mathbf{A}) \in \mathcal{B}_t$ where $t \geq 1$ is fixed. Then there is an algorithm deciding whether $M(\mathbf{A}) \in \mathcal{M}$ in time $O(n^3)$, and this algorithm can be constructed from the given sentence(s) ϕ or ϕ_t for all t.

Remark. In the language of parametrized complexity, Theorem 4.1 says that the class of $\mathbb{F}$-represented matroids defined by MSO sentences ϕ or ϕ_t is fixed-parameter tractable with respect to the combined parameter $\langle \mathbb{F}, t\rangle$. Moreover, in the case (a), or in the case (b) when the sentences ϕ_k are constructible by an algorithm, the class $\mathcal{M}$ is uniformly fixed-parameter tractable.

So it follows that the properties of being Hamiltonian or a paving matroid can be efficiently recognized on $\mathbb{F}$-represented matroids of bounded branch-width. Other simple matroidal properties definable in the MSO logic are, for example, the properties of being identically self-dual, or being a "free spike" [11]. Moreover, all properties definable in the extended MSO theory of graphs (MS_2) are also MSO-definable over graphic matroids [8]. Several more interesting classical matroid properties are shown to be MSO-definable in the next sections.

5 Minor-Closed Properties

It is easy to see that the class of $\mathbb{F}$-representable matroids is minor-closed, and so is the class $\mathcal{B}_t$ of matroids of branch-width at most t. We say that a set S is *well-quasi-ordered* (WQO) if there are neither infinite antichains nor infinite strictly descending chains in S. By a deep result of [6], matroids of bounded branch-width which are representable over a fixed finite field $\mathbb{F}$ are WQO in the minor order. (However, unlike graphs, matroids are not WQO in general.) So it follows that any minor-closed matroid family $\mathcal{M}$ has a finite number of $\mathbb{F}$-representable excluded minors in $\mathcal{B}_t$. We now show that presence of one particular minor can be described by an MSO sentence.

Lemma 5.1. *Let N be a matroid. There is a (computable) MSO sentence ψ_N such that ψ_N is true on a matroid M if and only if M has an N-minor.*

Proof. N is a minor of M if and only if there are two sets C, D such that C is independent and D is coindependent in M, and that $N = M \setminus D/C$. Suppose that $N = M \setminus D/C$ holds. Then a set $X \subseteq E(N)$ is dependent in N if and only if there is a dependent set $Y \subseteq E(M)$ in M such that $Y - X \subseteq C$. (This simple claim may be more obvious when viewed over the dual matroid M^* — a set is dependent in M iff it intersects each basis of M^*, and $N^* = M^*/D \setminus C$.)

Since N is fixed, we may identify the elements of the (supposed) N-minor in M by variables $x_1, \ldots, x_n$ in order, where $n = |E(N)|$. Then, knowing the contract set C (and implicit D), we are able to say which subsets of $\{x_1, \ldots, x_n\}$ are dependent in $M \setminus D/C$. For each $J \subseteq [1, n]$, we write

$$mdep(x_j : j \in J; C) \equiv \exists Y \left[\neg\, indep(Y) \wedge \forall y \left(y \notin Y \vee y \in C \vee \bigvee\nolimits_{j \in J} y = x_j \right)\right].$$

Now, $M \setminus D/C$ is isomorphic to N iff the dependent subsets of $\{x_1, \ldots, x_n\}$ exactly match the dependent sets of N. Hence we express ψ_N as

$$\exists C\, \exists x_1, \ldots, x_n \left[\bigwedge\nolimits_{J \in \mathcal{J}_+} \neg\, mdep(x_j : j \in J; C) \wedge \bigwedge\nolimits_{J \in \mathcal{J}_-} mdep(x_j : j \in J; C)\right],$$

where $\mathcal{J}_+$ is the set of all $J \subseteq [1, n]$ such that $\{x_j : j \in J\}$ actually is independent in N, and where $\mathcal{J}_-$ is the complement of $\mathcal{J}_+$. □

Hence, in connection with Theorem 4.1, we conclude:

Theorem 5.2. *Let $t \geq 1$ be fixed, let $\mathbb{F}$ be a finite field, and let $\mathcal{M}$ be a minor-closed family. Given a matrix $\boldsymbol{A}$ over $\mathbb{F}$ with n columns such that $M(\boldsymbol{A}) \in \mathcal{B}_t$, one can decide whether the matroid $M(\boldsymbol{A})$ belongs to $\mathcal{M}$ in time $O(n^3)$.*

Proof. As already noted above, the family $\mathcal{M}$ has a finite number of $\mathbb{F}$-representable excluded minors $X_1, \ldots, X_p \in \mathcal{B}_t$. Keeping in mind that all minors of $M(\boldsymbol{A})$ also belong to $\mathcal{B}_t$, we see that $M(\boldsymbol{A}) \in \mathcal{M}$ iff $M(\boldsymbol{A})$ has no minors isomorphic to $X_1, \ldots, X_p$. (For formal completeness, we may verify $M(\boldsymbol{A}) \in \mathcal{B}_t$ using Corollary 3.2.) We write $\phi_t \equiv \neg\psi_{X_1} \wedge \ldots \wedge \neg\psi_{X_p}$ using Lemma 5.1. Finally, we apply Theorem 4.1(b). □

Applications of this theorem include determining the exact branch-width (cf. Section 3) or tree-width of a matroid, or deciding matroid orientability and representability over another field.

Remark. Unfortunately, the proof of Theorem 5.2 is non-constructive — there is no way in general how to compute the excluded minors $X_1, \ldots, X_p$, not even their number or size. So we cannot speak about uniform fixed-parameter tractability here.

6 Matroid Connectivity

Another interesting task is to describe matroid connectivity in the MSO logic. That can be done quite easily.

Lemma 6.1. *Let M be a matroid on the ground set E, and let $k \geq 1$. There is an MSO formula $\sigma_k(X)$ which is true for $X \subseteq E$ if and only if $\lambda_M(X) \geq k+1$.*

Proof. By definition, $\lambda_M(X) \geq k+1$ iff $\mathrm{r}_M(X) + \mathrm{r}_M(E-X) \geq \mathrm{r}(M) + k$. Using standard matroidal arguments, this is equivalent to stating that there exist two bases B_1, B_2 of M such that $B_2 \cap X \subset B_1$ and $|(B_1 - B_2) \cap X| \geq k$. We may formalize this statement as

$$\sigma_k(X) \equiv \exists B_1, B_2 \Big[\mathit{basis}(B_1) \wedge \mathit{basis}(B_2) \wedge \forall x \big((x \in B_2 \wedge x \in X) \to x \in B_1 \big) \wedge$$

$$\wedge\, \exists z_1, \ldots, z_k \Big(\bigwedge_{i \neq j} z_i \neq z_j \wedge \bigwedge_i z_i \in X \wedge \bigwedge_i z_i \in B_1 \wedge \bigwedge_i z_i \notin B_2 \Big) \Big] . \quad \square$$

So we may finish this section with the next immediate result:

Corollary 6.2. *For each $n > 1$, there is an MSO sentence κ_n which is true on a matroid M if and only if M is n-connected.*

7 Transversal Matroids

A matroid M is *transversal* if there is a bipartite graph G with vertex parts $V = E(M)$ and W, such that the rank of any set X in M equals the largest size of a matching incident with X in G. (Equivalently, a transversal matroid is a union of rank-1 matroids.) We consider transversal matroids here mainly because they have long history of research, but there is not much known about their relation to branch-width.

Two elements e, f in a matroid M are *parallel* if $\{e, f\}$ form a circuit, and e, f are *in series* if e, f are parallel in the dual M^*. A *series minor* of a matroid M is obtained by a sequence of contractions of series elements and arbitrary deletions of elements in M. A matroid having a representation over $GF(2)$ is called a *binary matroid*.

The trouble with transversal matroids is that these are not closed under taking minors or duals. However, series minors of transversal matroids are transversal again. We cannot use a "series" analogue of Theorem 5.2 since there is no well-quasi-ordering property of series minors even of bounded branch-width. Still, we can say a bit:

Theorem 7.1. *There is an MSO sentence τ which is true on a matroid M if and only if M is a binary transversal matroid.*

Sketch of proof. Let C_k^2 denote the graph obtained from a length-k cycle C_k by adding one parallel edge to each edge of C_k. According to [3], the following is true: A matroid M is both binary and transversal if and only if M has no series minor isomorphic to either the 4-element line $U_{2,4}$, or the graphic matroids $M(K_4)$ or $M(C_k^2)$ for $k \geq 3$.

Let $N = M \setminus D/C$ be a minor of M, and let $F = E(N)$. There are no problems to express that N is a series minor of M, i.e. that C consists of series elements of $M \setminus D$. (For simplicity, we assume no coloops.) We write

$$\forall x \in C\, \exists y \in F\, \forall Z\ \big[\big(Z \subseteq F \cup C \wedge \mathit{basis}(Z)\big) \rightarrow \big(x \in Z \vee y \in Z\big)\big]\ .$$

Now let P be a matroid. We may express whether P is isomorphic to $M(C_k^2)$ (regardless of the value of k) as follows

$$\exists Z\big[\, \mathit{circuit}(Z) \wedge \forall x \notin Z\, \exists y \in Z\, \big(\mathit{circuit}(x,y)\big) \wedge \forall y \in Z\, \exists! x\, \big(x \notin Z \wedge \mathit{circuit}(x,y)\big)\big]$$

where $\exists! x\, \Pi(x)$ is a shortcut for $\exists x\, \Pi(x) \wedge \forall x, x' \big(x = x' \vee \neg\Pi(x) \vee \neg\Pi(x')\big)$.

The rest of the proof proceeds by combining the previous formulas with the ideas in the proof of Lemma 5.1. (Considering matroid P as a minor of M, we use the predicate *mdep* from that proof to express *circuit* in the above formula.) We leave technical details to the reader. □

Since the proof of Theorem 7.1 is very specific to binary matroids, we doubt that it could be extended to all matroids. Thus we ask:

Problem 7.2. Is the property of being a transversal matroid MSO-definable?

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