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Are matroids interesting combinatorial structures?

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1 Motivation

The Graph Minor Project
[Robertson and Seymour, 80's – 90's], [others later. . .]

- Formalized the notions of tree-width and branch-width (similar notions).
- Proved *Wagner's conjecture* – WQO property of graph minors.
  (Among the partial steps: WQO of graphs of bounded tree-width, excluded grid theorem, description of graphs excluding a complete minor.)
- Testing for an arbitrary fixed graph *minor in cubic time*.

Tree-like Graphs and Logic

- [Seese, 1975] Undecidability of an *MSO theory of large grids*.
- [Courcelle, 1988] Decidability of an *MSO theory* of graphs: The class of all (finite) graphs of bounded tree-width has decidable MS_2 theory.
  (Independently by [Arnborg, Lagergren, and Seese, 1991].)
- [Seese, 1991] Decidability of the MS_2 theory *implies bounded tree-width*.

Results closely related to *linear-time algorithms* on bounded tree-width graphs.

Current Trends in Matroids

- [Geelen, Gerards, Robertson, Whittle, and ..., late 90's – future] Extending the *ideas of graph minors* to matroids (over finite fields).
  (For example: WQO of matroids of bounded branch-width (over finite fields), excluded grid theorem, other technical results...)
- [PH, 2002] *Decidability for matroids*: The class of all $GF(q)$-representable matroids of bounded branch-width has a decidable MSO theory.
- [Seese and PH, 2004] Decidability of the matroidal MSO theory *implies a bounded branch-width*.

The new issue – Clique-Width

- [Courcelle et al, 1993] The definition (constructing a graph using a bounded number of labels).
- [Courcelle, Makowsky, Rotics, 2000] Decidability of the MS_1 theory.
- [Oum and Seymour, 2003] An approximation of graph clique-width via *rank-width*, which actually is a *matroid branch-width*.

Hence, we see an influence in both ways: *graph* $\leftrightarrow$ *matroid* theories.

2 Basics of Matroids

A matroid is a pair $M = (E, \mathcal{B})$ where

- $E = E(M)$ is the *ground set* of M (elements of M),
- $\mathcal{B} \subseteq 2^E$ is a collection of *bases* of M,
- the bases satisfy the "exchange axiom"
  $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 - B_2$,
  $\exists y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

Otherwise, a *matroid* is a pair $M = (E, \mathcal{I})$ where

- $\mathcal{I} \subseteq 2^E$ is the collection of *independent sets* (subsets of bases) of M.

The definition was inspired by an abstract view of *independence* in linear algebra and in combinatorics [Whitney, Birkhoff, Tutte,. . .].

Notice exponential amount of information carried by a matroid.

Literature: J. Oxley, Matroid Theory, Oxford University Press 1992,1997.

Some **elementary matroid terms** are

- *independent set* ≈ a subset of some basis,
  dependent set ≈ not independent,
- *circuit* ≈ a minimal dependent set of elements,
  triangle ≈ a circuit on 3 elements,
- *hyperplane* ≈ a maximal set containing no basis,
  cocircuit ≈ the complement of a hyperplane,
- *rank function* ≈ "*dimension*" of X,
  $\mathrm{r}_M(X)$ = maximal size of an M-independent subset $I_X \subseteq X$.

(Notation is taken from linear algebra and from graph theory...)

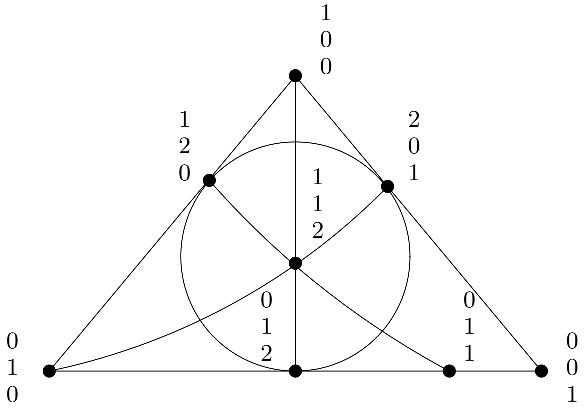
Axiomatic descriptions of matroids via independent sets, circuits, hyperplanes, or rank function are possible, and often used.

Vector matroid — a straightforward motivation:

- Elements are vectors over $\mathbb{F}$,
- independence is usual linear independence,
- the vectors are considered as columns of a matrix $\boldsymbol{A} \in \mathbb{F}^{r \times n}$.
  ($\boldsymbol{A}$ is called a *representation* of the matroid $M(\boldsymbol{A})$ over $\mathbb{F}$.)

Not all matroids are vector matroids.

An example of a rank-3 vector matroid with 8 elements over $GF(3)$:

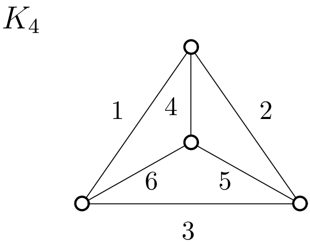


Graphic matroid $M(G)$ — the combinatorial link:

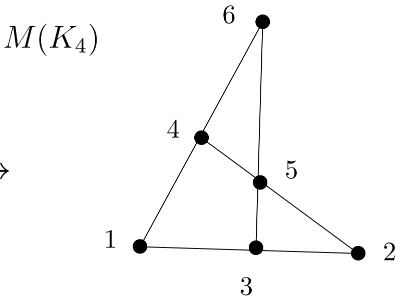
- Elements are the edges of a graph,
- independence $\sim$ acyclic edge subsets,
- bases $\sim$ spanning (maximal) forests,
- circuits $\sim$ graph cycles,
- the rank function $\mathrm{r}_M(X)$ = the number of vertices minus the number of components induced by X.

Only few matroids are graphic, but all *graphic ones are vector matroids* over any field.

Example:



$\longrightarrow$



3 MSO Theories

MSOL – monadic second-order logic:
propositional + quantification over elements and sets.

MSOL + class of structures $\Longrightarrow$ **MSO theory** (of the structures).

For graphs

- *Adjacency graphs* – formed by vertices and an adjacency relation.
  $\rightarrow$ MS_1 theory
- *Incidence graphs* – formed by vertices and edges (two-sorted structure), with an incidence relation.
  $\rightarrow$ MS_2 theory (A stronger language than MS_1.)

For matroids?

- The ground set $E(M)$, and what relation?
  No FO predicate is enough to describe all matroids! (An easy counting argument.)
- So using a set predicate to describe matroid structure. . .

Matroidal MSO Theory

A **matroid** *in logic* – the ground set $E = E(M)$ with all subsets 2^E,
– and a predicate *indep* on 2^E, s.t. *indep*(F) iff $F \subseteq E$ is independent.

The *MSO theory of matroids* – language of MSOL applied to such matroids.
($\rightarrow$ MS_M theory)

Basic expressions:

- *basis*$(B) \equiv$ *indep*$(B) \wedge \forall D(B \not\subseteq D \vee B = D \vee \neg$ *indep*$(D))$

  A basis is a maximal independent set.

- *circuit*$(C) \equiv \neg$ *indep*$(C) \wedge \forall D(D \not\subseteq C \vee D = C \vee$ *indep*$(D))$

  A circuit C is dependent, but all proper subsets of C are independent.

- *cocircuit*$(C) \equiv \forall B[$ *basis*$(B) \rightarrow \exists x(x \in B \wedge x \in C)] \wedge$
  $\wedge \forall X[X \not\subseteq C \vee X = C \vee \exists B($ *basis*$(B) \wedge \forall x(x \notin B \vee x \notin X))]$

  A cocircuit C (a dual circuit) intersects every basis, but each proper subset of C is disjoint from some basis.

How strong is the matroidal MSO language?

Expressive Power of Matroid MSO

[PH,2002–2004]

Defining a graph via its cycle matroid:

- The matroids of all trees of the same size are isomorphic.
- Even more troubles with loops.
- A matroidal circuit has no order of elements on it, unlike a graph cycle. (Cf. the dual – a parallel class of graph edges.)
- One has to require 3-connectivity to fully describe the underlying graph in terms of its cycle matroid!

Defining MS_2 properties in the corresponding cycle-matroid MSO:

- Any MS_2 sentence about a loopless 3-connected graph G can be formulated as an MS_M sentence about the cycle matroid $M(G)$.
- For less-connected graphs G, use the graph $G \uplus K_3$ (adding 3 more vertices connected to everything).
- Conversely, edge-set independence is MS_2 definable.

4 More on Matroids

Remark. About matroids on an input:

To describe an n-element matroid, one has to specify properties of all 2^n subsets. So giving a complete description on the input would *ruin any complexity measures*.

Solutions:

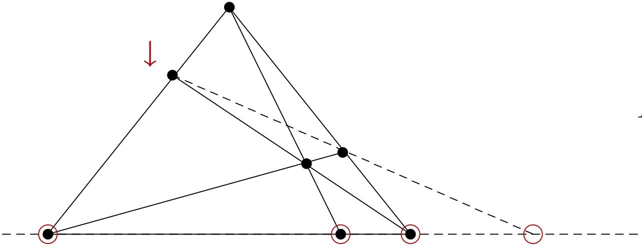
- Give a matroid via a *rank oracle* – answering queries about the rank.
- Give a special matroid with a particular *small representation*. (Likewise a matrix for a vector matroid.)

Matroid duality M^* (exchanging bases with their complements) $\sim$ topological duality in planar graphs, or transposition of the standard-form (i.e. without some basis) matrices.

Element deletion $\sim$ usual deletion of a graph edge or a vector.

Element contraction (corresponds to deletion in the dual matroid) $\sim$ edge contraction in a graph, or projection of the matroid from a vector (i.e. a linear transformation having a kernel formed by this vector).

Matroid minor — obtained by a sequence of element deletions and contractions, order of which does not matter.



$F_7^- \to U_{2,4}$

Example – MSO minor testing

Lemma 4.1. *For each fixed matroid N; a (computable) MSO sentence ψ_N telling us whether there is an N-minor.*

Proof: The sentence ψ_N over a matroid M:

- identify the elements of the (supposed) N-minor in M by variables $x_1, \ldots, x_n$ in order, where $n = |E(N)|$,
- assuming the contract-set C (implicit del.-set $D = E(M) - C - \{x_1, \ldots, x_n\}$), describe dependency in the minor $M \setminus D/C$:

$$\textit{minor-dep}(x_j : j \in J; C) \equiv$$

$$\exists Y \left[\neg\, \textit{indep}(Y) \wedge \forall y \in Y \left(y \in C \vee \bigvee\nolimits_{j \in J} y = x_j \right)\right],$$

- now, $M \setminus D/C$ is isomorphic to N iff dependency in $\{x_1, \ldots, x_n\}$ matches dependency in N; hence

$$\psi_N \equiv \exists C\, \exists x_1, \ldots, x_n$$

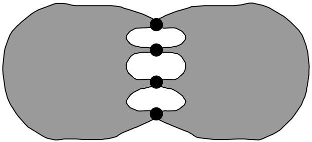
$$\left[\bigwedge\nolimits_{J \in \mathcal{J}_-} \textit{minor-dep}(x_j : j \in J; C) \wedge \bigwedge\nolimits_{J \in \mathcal{J}_+} \neg\, \textit{minor-dep}(x_j : j \in J; C)\right],$$

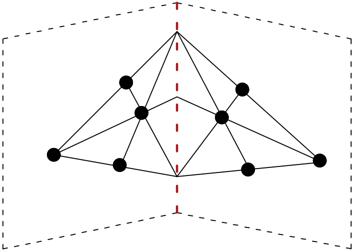
where $\mathcal{J}_+$ are the independent index-sets in $2^{[1,n]}$ of N, and $\mathcal{J}_-$ the complement.

Matroid Connectivity – an alternative view of graph connectivity

Connectivity function $\lambda_G(X)$ = number of vertices in G
incident both with edges of X and of $E(G) - X$.

A 4-separation in a graph:





A 3-separation in a matroid:

Matroid connectivity $\underline{\lambda_M(X) = \mathrm{r}_M(X) + \mathrm{r}_M(E - X) - \mathrm{r}(M) + 1}$

(geometrically the "rank of spans intersection" $\langle X \rangle \cap \langle E - X \rangle$ plus 1).

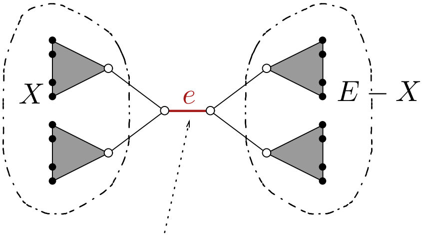
A *k-separation* $(X, E - X)$: $\lambda(X) \leq k$ and $|X|, |E - X| \geq k$.

Then, **high connectivity ≈ no small separations**.

5 Branch-Width

Graphs or matroids (or arb. sym. submod. λ) $\longrightarrow$ a **branch decomposition:**

- Decomposed to a *cubic tree* (degrees ≤ 3), and
- edges / elements mapped one-to-one to the tree leaves (with no reference to graph vertices).
- Tree edges have *width* as follows:



width$(e) = \lambda(X)$ where X is *"displayed" by* e in the tree.

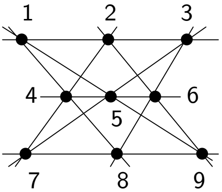
(Using graph connectivity $\lambda_G()$, or matroid connectivity $\lambda_M()$, resp.)

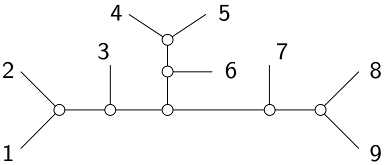
Branch-width = min. of max. edge widths over all decompositions.

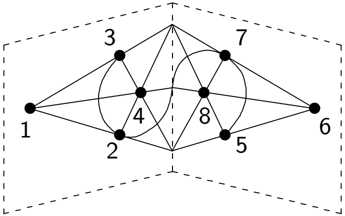
(Branch-width is within a constant factor of tree-width.)

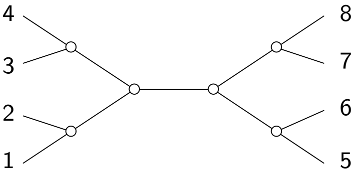
Branch decompositions of matroids

both of width 3:









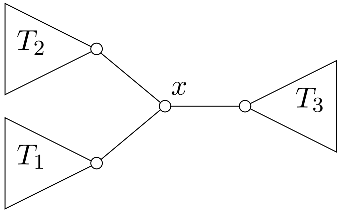
BTW, a Matroid Tree-Width

(First suggestion by [Geelen, unpublished], modified [PH and Whittle, 2003].)

A **tree decomposition of a matroid** M is (T, τ), where

- T a tree, and $\tau : E(M) \to V(T)$ an arbitrary mapping
  (nothing like the "bags"!),
- *width of a node* x in T is as follows:
  let $T_1, \dots, T_d$ be the connected components of $T - x$, then

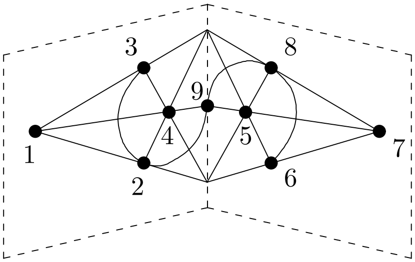
$$\text{width of } x = \sum_{i=1}^{d} \mathrm{r}_M \Big(E(M) - \tau^{-1}(V(T_i)) \Big) - (d-1) \cdot \mathrm{r}(M) \,.$$

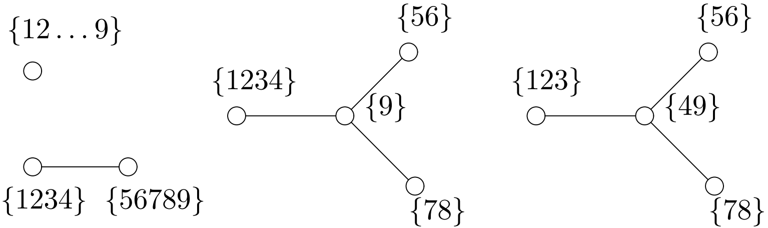
$\tau : E(M) \to$ 

Tree-width of M = min. of max. node widths over all decompositions.

(This parameter equals usual tree-width on graphic matroids!)

Tree decompositions of matroids





widths: 4,3 3 4

6 Computability and decidability on matroids

Considering matroids represented over a finite field $\mathbb{F}$.

Transformation: A matroid M and a branch decomposition $\rightarrow$
a **parse tree** $\bar{T}$ for $M = P(\bar{T})$.

[PH,2002] Computable in cubic FPT time for matroids of bounded branch-width over $\mathbb{F}$ (no branch decomposition required, approx. factor 3).

Theorem 6.1. [PH] *Let $t \geq 1$, and ϕ be a sentence in matr. MSOL. Then there exists a (constructible) finite tree automaton $\mathcal{A}_t^\phi$ accepting those parse trees for matroids over $\mathbb{F}$ which posses ϕ, i.e. those $\bar{T}$ such that $P(\bar{T}) \models \phi$.*

This result, together with an algorithm constructing the parse tree, provides an efficient way to verify MSO-definable properties over matroids of bounded branch-width.

Corollary 6.2. *If $\mathcal{B}_t$ is the class of all matroids representable over $\mathbb{F}$ of branch-width at most t, then the theory $\mathrm{Th}_{MSO}(\mathcal{B}_t)$ is decidable.*

Sketch: It is enough to verify emptiness of the complementary automaton $\neg\mathcal{A}_t^\phi$ over all valid parse trees.

A new result, cf. the talk of D. Seese:

Theorem 6.3. [Seese and PH, 2004] *Let $\mathcal{N}$ be a class of matroids that are representable by matrices over $\mathbb{F}$. If the monadic second-order theory* $\mathrm{Th}_{MSO}(\mathcal{N})$ *is decidable, then the class $\mathcal{N}$ has bounded branch-width.*

(Analogous to a result of [Seese, 1991] on the MS_2 theory of graphs.)