On decidability of MSO theories of combinatorial structures:

Towards general matroids?

Petr Hliněný

Faculty of Informatics, Masaryk University
Botanická 68a, 602 00 Brno, Czech Rep.

e-mail: hlineny@fi.muni.cz
http://www.fi.muni.cz/~hlineny

(Parts based on joint work with **Detlef Seese**, University Karlsruhe TH)

1 Motivation

The Graph Minor Project [Robertson and Seymour]

- Proved *Wagner's conjecture* – WQO property of graph minors.
  (Among the partial steps: WQO of graphs of bounded *tree-width*, excluded *grid* theorem, description of graphs excluding a complete minor.)
- Testing for an arbitrary fixed graph minor in cubic time.

1 Motivation

The Graph Minor Project [Robertson and Seymour]

- Proved *Wagner's conjecture* – WQO property of graph minors.
  (Among the partial steps: WQO of graphs of bounded *tree-width*, excluded *grid* theorem, description of graphs excluding a complete minor.)
- Testing for an arbitrary fixed graph minor in cubic time.

Tree-like Graphs and Logic

- [Seese, 1975] Undecidability of the *MSO theory of square grids*.
- [Courcelle, 1988] Decidability of the MSO theory of graphs: The class of all (finite) graphs of bounded tree-width has decidable MS_2 theory.
- [Seese, 1991] Decidability of the MS_2 theory implies bounded tree-width.
- [Courcelle et al, 1993] The definition of *clique-width* (constructing a graph using a bounded number of labels).
  [Courcelle, Makowsky, Rotics, 2000] Decidability of the MS_1 theory.
- [Oum and Seymour, 2003] *Rank-width* to approximate clique-width.
  This notion has a strong matroidal essence !

2 An Automata-based Approach

Separations and parse trees

- Conside "combinatorial" structures with distinguished *boundaries*.

  The boundaries are used to glue two substructures together, such that all "possible interference" between those two happens on their boundaries.

2 An Automata-based Approach

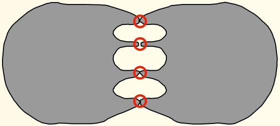
Separations and parse trees

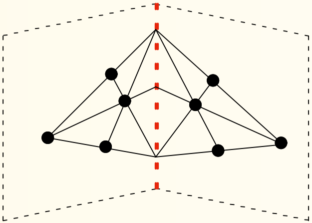
- Conside "combinatorial" structures with distinguished *boundaries*.

  The boundaries are used to glue two substructures together, such that all "possible interference" between those two happens on their boundaries.

- $\mapsto$ leading to *separations* (of the ground set) and their *guts* (the common boundary), with natural meaning on graphs (and matroids).

4-separation in a graph





3-separation in a matroid

2 An Automata-based Approach

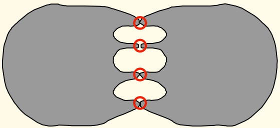
Separations and parse trees

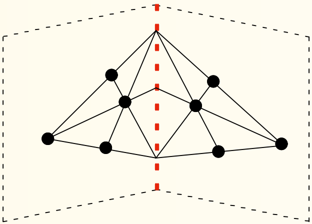
- Conside "combinatorial" structures with distinguished *boundaries*.

  The boundaries are used to glue two substructures together, such that all "possible interference" between those two happens on their boundaries.

- $\mapsto$ leading to *separations* (of the ground set) and their *guts* (the common boundary), with natural meaning on graphs (and matroids).

4-separation in a graph





3-separation in a matroid

- *Parse trees*: The (above) boundary-glue operation is used to "*build*" a structures from smaller boundaried pieces in a tree-like fashion.

Properties decidable by automata

Question: When a *property* ϕ can be tested by a finite tree automaton running on the (above) parse trees?

- Using a "localization" of the **Myhill-Nerode theorem**:
  - An approach originally suggested by Abrahamson and Fellows.

Properties decidable by automata

Question: When a *property* ϕ can be tested by a finite tree automaton running on the (above) parse trees?

- Using a "localization" of the **Myhill-Nerode theorem**:
  - An approach originally suggested by Abrahamson and Fellows.
  - Define an *equivalence* $\approx_\phi$ on the class of boundaried *struct.* $\mathcal{C}_k$; $A, B \in \mathcal{C}_k$, $\boldsymbol{A \approx_\phi B}$ if and only if

$$\underline{\forall \boldsymbol{D} \in \boldsymbol{\mathcal{C}_k} : \; \boldsymbol{A} \oplus \boldsymbol{D} \models \boldsymbol{\phi} \iff \boldsymbol{B} \oplus \boldsymbol{D} \models \boldsymbol{\phi}}.$$

    ($A \approx_\phi B$ – carrying the same info. about ϕ on their boundaries.)

Properties decidable by automata

Question: When a *property* ϕ can be tested by a finite tree automaton running on the (above) parse trees?

- Using a "localization" of the **Myhill-Nerode theorem**:
  - An approach originally suggested by Abrahamson and Fellows.
  - Define an *equivalence* $\approx_\phi$ on the class of boundaried *struct.* $\mathcal{C}_k$; $A, B \in \mathcal{C}_k$, $\boldsymbol{A \approx_\phi B}$ if and only if

$$\underline{\forall D \in \mathcal{C}_k : \; A \oplus D \models \phi \iff B \oplus D \models \phi}.$$

    ($A \approx_\phi B$ – carrying the same info. about ϕ on their boundaries.)

- **(Meta)Theorem 1.**
  For fixed k, there is a finite tree automaton $\mathcal{A}_{\phi,k}$ accepting precisely those parse trees of width k (of structures from $\mathcal{C}_k$) that posses property ϕ, if and only if the equivalence $\approx_\phi$ has finite index over $\mathcal{C}_k$.

  Beware that this meta-statement needs a specific proof in each case(!); for instance, it is not straightforwardly true for graph clique-width.

Straightforward applications

- Graphs (MSO_2) of bounded branch-width.

  (Although Abrahamson and Fellows applied that first to graphs of bounded tree-width, that was quite complicated and unnatural. . .)

- Matroids (MSO) of bounded branch-width which are represented over a finite field.

- Graphs (MSO_1) of bounded rank-width.

3 Basics of Matroids

A matroid is a pair $M = (E, \mathcal{B})$ where

- $E = E(M)$ is the *ground set* of M (elements of M),
- $\mathcal{B} \subseteq 2^E$ is a collection of *bases* of M,
- the bases satisfy the "exchange axiom"
  $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 - B_2$,
  $\exists y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

3 Basics of Matroids

A matroid is a pair $M = (E, \mathcal{B})$ where

- $E = E(M)$ is the *ground set* of M (elements of M),
- $\mathcal{B} \subseteq 2^E$ is a collection of *bases* of M,
- the bases satisfy the "exchange axiom"
  $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 - B_2$,
  $\exists y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

Otherwise, a *matroid* is a pair $M = (E, \mathcal{I})$ where

- $\mathcal{I} \subseteq 2^E$ is the collection of *independent sets* (subsets of bases) of M.

3 Basics of Matroids

A matroid is a pair $M = (E, \mathcal{B})$ where

- $E = E(M)$ is the *ground set* of M (elements of M),
- $\mathcal{B} \subseteq 2^E$ is a collection of *bases* of M,
- the bases satisfy the "exchange axiom"
  $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 - B_2$,
  $\exists y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

Otherwise, a *matroid* is a pair $M = (E, \mathcal{I})$ where

- $\mathcal{I} \subseteq 2^E$ is the collection of *independent sets* (subsets of bases) of M.

The definition was inspired by an abstract view of *independence* in linear algebra and in combinatorics [Whitney, Birkhoff, Tutte,...].

Notice exponential amount of information carried by a matroid.

Literature: J. Oxley, Matroid Theory, Oxford University Press 1992,1997.

Some **elementary matroid terms** are

- *independent set* ≈ a subset of some basis,
  dependent set ≈ not independent,
- *circuit* ≈ a minimal dependent set of elements,
  triangle ≈ a circuit on 3 elements,
- *hyperplane* ≈ a maximal set containing no basis,
  cocircuit ≈ the complement of a hyperplane,

Some **elementary matroid terms** are

- *independent set* $\approx$ a subset of some basis,
  dependent set $\approx$ not independent,
- *circuit* $\approx$ a minimal dependent set of elements,
  triangle $\approx$ a circuit on 3 elements,
- *hyperplane* $\approx$ a maximal set containing no basis,
  cocircuit $\approx$ the complement of a hyperplane,
- *rank function* $\approx$ "*dimension*" of X,

  $\mathrm{r}_M(X) =$ maximal size of an M-independent subset $I_X \subseteq X$.

Some **elementary matroid terms** are

- *independent set* ≈ a subset of some basis,
  dependent set ≈ not independent,
- *circuit* ≈ a minimal dependent set of elements,
  triangle ≈ a circuit on 3 elements,
- *hyperplane* ≈ a maximal set containing no basis,
  cocircuit ≈ the complement of a hyperplane,
- *rank function* ≈ "*dimension*" of X,
  $\mathrm{r}_M(X)$ = maximal size of an M-independent subset $I_X \subseteq X$.
- *connectivity function* ≈ like "connecting paths" between two sides of a separation (cut) in a graph,
  $\lambda_M(X) = \mathrm{r}_M(X) + \mathrm{r}_M(E - X) - \mathrm{r}(M) + 1$ (= guts rank + 1).

Notation taken from linear algebra and from graph theory...

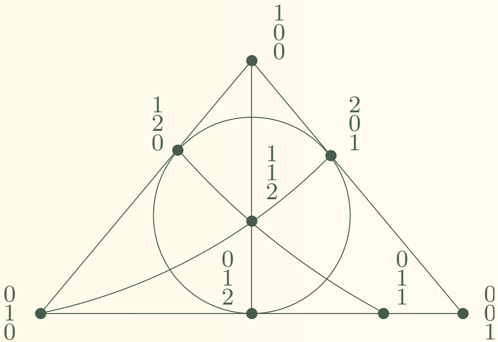
Axiomatic descriptions of matroids via independent sets, circuits, hyperplanes, or rank function are possible, and often used.

Vector matroid — a straightforward motivation:

- Elements are vectors over $\mathbb{F}$,
- independence is usual linear independence,
- the vectors are considered as columns of a matrix $\boldsymbol{A} \in \mathbb{F}^{r \times n}$.
  ($\boldsymbol{A}$ is called a *representation* of the matroid $M(\boldsymbol{A})$ over $\mathbb{F}$.)

Not all matroids are vector matroids.

An **example** of a rank-3 vector matroid with 8 elements over $GF(3)$:



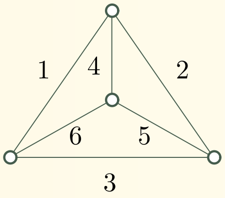
Graphic matroid $M(G)$ — the combinatorial link:

- Elements are the edges of a graph,
- independence $\sim$ acyclic edge subsets,
- bases $\sim$ spanning (maximal) forests,
- circuits $\sim$ graph cycles,
- the rank function $\mathrm{r}_M(X) =$ the number of vertices minus the number of components induced by X.

Only few matroids are graphic, but all *graphic ones are vector matroids* over any field.

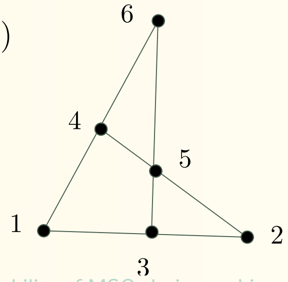
Example:

K_4



$\longrightarrow$

$M(K_4)$



Branch-width

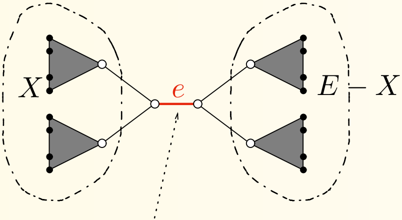
Graphs or matroids (or arb. sym. submod. λ) $\longrightarrow$ a **branch decomposition:**

- Decomposed to a *sub-cubic tree* (degrees ≤ 3), and
- edges / elements mapped one-to-one to the tree leaves (with no reference to graph vertices).

Branch-width

Graphs or matroids (or arb. sym. submod. λ) $\longrightarrow$ a **branch decomposition:**

- Decomposed to a *sub-cubic tree* (degrees ≤ 3), and
- edges / elements mapped one-to-one to the tree leaves (with no reference to graph vertices).
- Tree edges have *width* as follows:



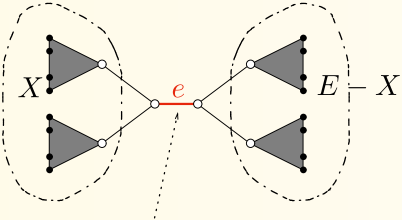
$\underline{\text{width}(e) = \lambda(X)}$ where X is *"displayed" by* e in the tree.

(Using graph connectivity $\lambda_G()$, or matroid connectivity $\lambda_M()$, resp.)

Branch-width

Graphs or matroids (or arb. sym. submod. λ) $\longrightarrow$ a **branch decomposition:**

- Decomposed to a *sub-cubic tree* (degrees ≤ 3), and
- edges / elements mapped one-to-one to the tree leaves (with no reference to graph vertices).
- Tree edges have *width* as follows:



$\underline{\text{width}(e) = \lambda(X)}$ where X is *"displayed" by* e in the tree.

(Using graph connectivity $\lambda_G()$, or matroid connectivity $\lambda_M()$, resp.)

Branch-width = min. of max. edge widths over all decompositions.
(Branch-width is within a constant factor of tree-width.)

4 Matroidal MSO Theory

A **matroid** in logic – the ground set $E = E(M)$ with all subsets 2^E,

– and a predicate *indep* on 2^E, s.t. *indep*(F) iff $F \subseteq E$ is independent.

The *MSO theory of matroids* – language of MSOL applied to such matroids.

4 Matroidal MSO Theory

A **matroid** in logic – the ground set $E = E(M)$ with all subsets 2^E,

– and a predicate *indep* on 2^E, s.t. *indep*(F) iff $F \subseteq E$ is independent.

The *MSO theory of matroids* – language of MSOL applied to such matroids.

Basic expressions:

- *basis*$(B) \equiv$ *indep*$(B) \wedge \forall D(B \not\subseteq D \vee B = D \vee \neg$ *indep*$(D))$
  A basis is a maximal independent set.
- *circuit*$(C) \equiv \neg$ *indep*$(C) \wedge \forall D(D \not\subseteq C \vee D = C \vee$ *indep*$(D))$
  A circuit C is dependent, but all proper subsets of C are independent.
- *cocircuit*$(C) \equiv \forall B[$ *basis*$(B) \rightarrow \exists x(x \in B \wedge x \in C)] \wedge$
  $\wedge \forall X[X \not\subseteq C \vee X = C \vee \exists B($ *basis*$(B) \wedge \forall x(x \notin B \vee x \notin X))]$
  A cocircuit C (a dual circuit) intersects every basis, but each proper subset of C is disjoint from some basis.

4 Matroidal MSO Theory

A **matroid** in logic – the ground set $E = E(M)$ with all subsets 2^E,
– and a predicate *indep* on 2^E, s.t. *indep*(F) iff $F \subseteq E$ is independent.

The *MSO theory of matroids* – language of MSOL applied to such matroids.

Basic expressions:

- $\mathit{basis}(B) \equiv \mathit{indep}(B) \wedge \forall D (B \not\subseteq D \vee B = D \vee \neg\, \mathit{indep}(D))$
  A basis is a maximal independent set.
- $\mathit{circuit}(C) \equiv \neg\, \mathit{indep}(C) \wedge \forall D (D \not\subseteq C \vee D = C \vee \mathit{indep}(D))$
  A circuit C is dependent, but all proper subsets of C are independent.
- $\mathit{cocircuit}(C) \equiv \forall B [\, \mathit{basis}(B) \rightarrow \exists x (x \in B \wedge x \in C)] \wedge$
  $\wedge \forall X [X \not\subseteq C \vee X = C \vee \exists B (\, \mathit{basis}(B) \wedge \forall x (x \notin B \vee x \notin X))]$
  A cocircuit C (a dual circuit) intersects every basis, but each proper subset of C is disjoint from some basis.

How strong is the matroidal MSO language?

– neglecting low connectivity, (roughly) on the level of graph MSO_2.

Decidability on matroids

Considering matroids represented over a finite field $\mathbb{F}$.

Transformation: A matroid M over $\mathbb{F}$ and a branch decomposition $\mapsto$
a **parse tree** $\bar{T}$ for $M = P(\bar{T})$.

Theorem 2. [PH 2005] *The parse tree is computable in cubic FPT time for matroids of bounded branch-width over* $\mathbb{F}$.

(No branch decomp. required, approx. factor 3*. New [Oum, PH] optimally).*

Decidability on matroids

Considering matroids represented over a finite field $\mathbb{F}$.

Transformation: A matroid M over $\mathbb{F}$ and a branch decomposition $\mapsto$ a **parse tree** $\bar{T}$ for $M = P(\bar{T})$.

Theorem 2. [PH 2005] *The parse tree is computable in cubic FPT time for matroids of bounded branch-width over $\mathbb{F}$.*

(No branch decomp. required, approx. factor 3. New [Oum, PH] optimally).

The idea.

For a represented matroid, we declare a distinguished *subspace as a boundary*.

Bounded width $\Rightarrow$ fixed-rank finite geometry over $\mathbb{F}$ $\Rightarrow$ finite index of $\approx_\phi$ for every MSO sentence ϕ.

Decidability on matroids

Considering matroids represented over a finite field $\mathbb{F}$.

Transformation: A matroid M over $\mathbb{F}$ and a branch decomposition $\mapsto$
a **parse tree** $\bar{T}$ for $M = P(\bar{T})$.

Theorem 2. [PH 2005] *The parse tree is computable in cubic FPT time for matroids of bounded branch-width over $\mathbb{F}$.*

(No branch decomp. required, approx. factor 3. New [Oum, PH] optimally).

The idea.

For a represented matroid, we declare a distinguished *subspace as a boundary*.

Bounded width $\Rightarrow$ fixed-rank finite geometry over $\mathbb{F}$ $\Rightarrow$ finite index of $\approx_\phi$ for every MSO sentence ϕ.

Theorem 3. [PH 2003] *Let $t \geq 1$, and ϕ be a sentence in matr. MSOL. Then there exists a (constructible) finite tree automaton $\mathcal{A}_{\phi,t}$ accepting those parse trees $\bar{T}$ of width $\leq t$ for matroids over $\mathbb{F}$ such that $P(\bar{T}) \models \phi$.*

Corollary 4. *If $\mathcal{B}_t$ is the class of all matroids representable over $\mathbb{F}$ of branch-width at most t, then the theory $\mathsf{Th}_{MSO}(\mathcal{B}_t)$ is decidable.*

Corollary 4. *If $\mathcal{B}_t$ is the class of all matroids representable over $\mathbb{F}$ of branch-width at most t, then the theory $\mathsf{Th}_{MSO}(\mathcal{B}_t)$ is decidable.*

Complementing this statement, we have:

Theorem 5. [Seese and PH, 2005] *Let $\mathcal{N}$ be a class of matroids that are representable over $\mathbb{F}$. If the monadic second-order theory $\mathsf{Th}_{MSO}(\mathcal{N})$ is decidable, then the class $\mathcal{N}$ has bounded branch-width.*

Corollary 4. *If $\mathcal{B}_t$ is the class of all matroids representable over $\mathbb{F}$ of branch-width at most t, then the theory $\mathsf{Th}_{MSO}(\mathcal{B}_t)$ is decidable.*

Complementing this statement, we have:

Theorem 5. [Seese and PH, 2005] *Let $\mathcal{N}$ be a class of matroids that are representable over $\mathbb{F}$. If the monadic second-order theory $\mathsf{Th}_{MSO}(\mathcal{N})$ is decidable, then the class $\mathcal{N}$ has bounded branch-width.*

Why this idea does not generalize to all matroids?

Bounded width $\Rightarrow$ fixed-rank ~~finite~~ geometry $\not\Rightarrow$ finite index of $\approx_\phi$.

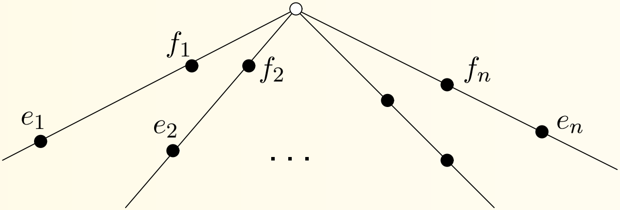
5 Some Undecidable Theories

- Of course, any class of matroids with unbounded grid minors. . .

5 Some Undecidable Theories

- Of course, any class of matroids with unbounded grid minors...
- [Seese and PH, 2005] The class of all *spikes*
  – special matroids of branch-width 3.

  This class interprets arbitrary grids via an easy encoding in *grid spikes*.

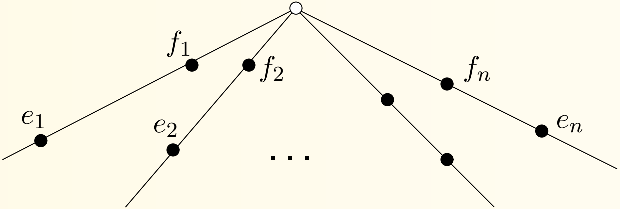


5 Some Undecidable Theories

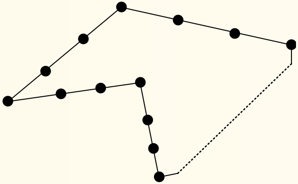
- Of course, any class of matroids with unbounded grid minors...

- [Seese and PH, 2005] The class of all *spikes*
  – special matroids of branch-width 3.

  This class interprets arbitrary grids via an easy encoding in *grid spikes*.



- A similar example with *swirls*...



- A striking example!

  (Thanks to a construction by [**Mayhew**, 2005]...)

The MSO theory of all rational matroids of rank 3 contains $\mathbf{MSO_1}$ of graphs.

- A striking example!

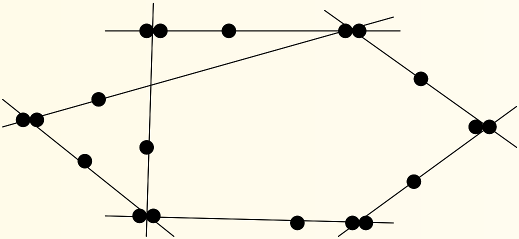
  (Thanks to a construction by [**Mayhew**, 2005]. . .)

  The MSO theory of all rational matroids of rank $\mathbf{3}$ contains $\mathbf{MSO_1}$ of graphs.

  Simple idea:

  – Interpret graph vertices as double-points in general position,

  – and place edges as single-points colinear with their endvertices.



6 Boundaries of MSO Decidability for matroids?

- We see much more strict condition must be imposed on general matroids to obtain decidable MSO theory.

  So what is the right matroidal “width” notion for this purpose?

6 Boundaries of MSO Decidability for matroids?

- We see much more strict condition must be imposed on general matroids to obtain decidable MSO theory.

  So what is the right matroidal "width" notion for this purpose?

- Possibly easier...

  What about studying the specific cases / subclasses (the class of spikes, the matroids of rank 3)? Are the presented structures the only "*forbidden substructures*" for MSO decidability?

  What "containment" relation (MSO-definable, of course) should we use here, is the *minor* relation good enough or shall we look for another one?

6 Boundaries of MSO Decidability for matroids?

- We see much more strict condition must be imposed on general matroids to obtain decidable MSO theory.

  So what is the right matroidal "width" notion for this purpose?

- Possibly easier...

  What about studying the specific cases / subclasses (the class of spikes, the matroids of rank 3)? Are the presented structures the only "*forbidden substructures*" for MSO decidability?

  What "containment" relation (MSO-definable, of course) should we use here, is the *minor* relation good enough or shall we look for another one?

- These interesting questions are subject of ongoing research...