A Note on Possible Extensions of Negami's Conjecture

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Abstract. A graph $\boldsymbol{H}$ is a cover of a graph $\boldsymbol{G}$ if there exists a mapping φ from $V(\boldsymbol{H})$ onto $V(\boldsymbol{G})$ such that for every vertex v of $\boldsymbol{G}$, φ maps the neighbours of v in $\boldsymbol{H}$ bijectively onto the neighbours of $\varphi(v)$ in $\boldsymbol{G}$. Negami conjectured in 1987 that a connected graph has a finite planar cover if and only if it embeds in the projective plane. This conjecture is not completely solved yet, but partial results due to Archdeacon, Fellows, Negami and the author are known.
This paper suggests another formulation of this conjecture that has a straightforward generalization to higher nonorientable surfaces, and provides some support for the generalized version.

1 Introduction

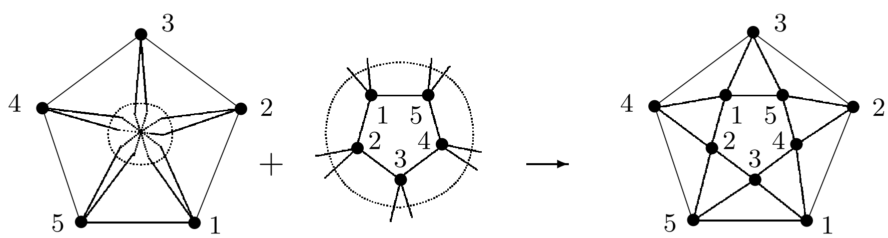

Fig. 1. A double planar cover of $\boldsymbol{K}_5$, constructed by lifting its projective drawing

We consider only *finite simple graphs* in this paper. A *surface* is a compact 2-manifold without boundary. We say that a graph $\boldsymbol{H}$ is a *cover* of a graph $\boldsymbol{G}$ if there exists a surjective mapping φ from $V(\boldsymbol{H})$ onto $V(\boldsymbol{G})$ such that for every vertex v of $\boldsymbol{G}$, φ maps the neighbours of v in $\boldsymbol{H}$ bijectively to the neighbours of $\varphi(v)$ in $\boldsymbol{G}$. If $\mathcal{S}$ is a surface, and the graph $\boldsymbol{G}$ has a cover that embeds on $\mathcal{S}$, then $\boldsymbol{G}$ has an $\mathcal{S}$*-cover.* We use the terms of *planar cover* (projective cover, Klein cover) in the obvious sense. Every planar graph has a planar cover by an identity mapping. As a non-trivial example, we mention a planar cover of the non-planar graph $\boldsymbol{K}_5$ (see Fig. 1), obtained by lifting a projective drawing into the universal covering space.

The previous construction of a double planar cover can be easily generalized for each projective planar graph; thus all projective planar graphs have finite planar covers. Negami proved in [8] that a connected graph has a double planar cover iff it embeds in the projective plane. He [9] also conjectured in general:

Conjecture 1. (S. Negami, 1987, "the 12∞-conjecture") *A connected graph has a finite planar cover if and only if it embeds in the projective plane.*

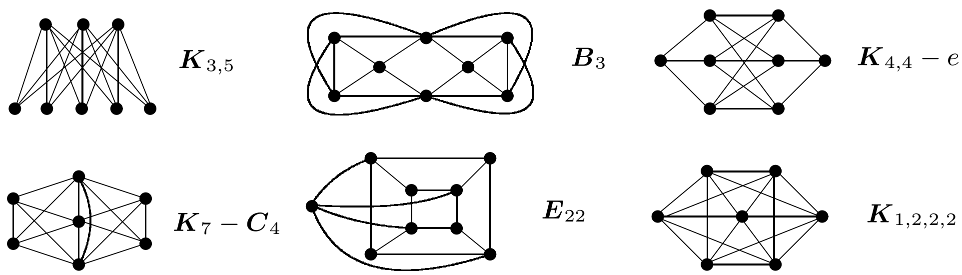

Fig. 2. Examples – 6 of the forbidden minors for the projective plane.

Note that the property of having a planar cover is hereditary under the minor ordering. So to prove the above stated conjecture, it is sufficient to prove that none of the minor-minimal nonprojective graphs [2] has a planar cover. The list contains 35 graphs, but it follows from work of Archdeacon (unpublished) that the problem can be reduced by $Y\Delta$-transformations to 11 graphs, see Fig. 2 for examples.

Soon after the conjecture was stated, Archdeacon [3], Fellows, and Negami [10] proved that 9 of these graphs have no planar cover. However, the other two graphs $\boldsymbol{K}_{4,4} - e$ and $\boldsymbol{K}_{1,2,2,2}$ remained unsolved for several years. The author [5] published last year a proof for the first one of the two remaining cases; so in order to prove the conjecture, it remains to show that the graph $\boldsymbol{K}_{1,2,2,2}$ has no finite planar cover. Recently, R. Thomas and the author [6] proved that, up to obvious constructions, there are at most 16 possible counterexamples to Negami's conjecture.

2 A generalization of the conjecture

The aim of this paper is to suggest another formulation of Conjecture 1 that has a straightforward generalization to other nonorientable surfaces.

Conjecture 2. *A connected graph embeds in the projective plane if and only if it has a projective cover.*

Proof *of equivalence with Conjecture 1.* It is enough to prove that a graph has a finite planar cover if and only if it has a finite projective cover. Indeed, a planar cover is a projective cover, too. On the other hand, let $\boldsymbol{H}$ be a projective cover of a graph $\boldsymbol{G}$, then $\boldsymbol{H}$ has a double planar cover $\boldsymbol{F}$ (obtained by lifting as in Fig. 1). One can check that the property of having a cover is a transitive relation, thus $\boldsymbol{F}$ is a planar cover of $\boldsymbol{G}$. ∎

The advantage of the latter formulation is that we speak about one surface only, and we directly relate to each other the properties of having a cover and of having an embedding in the surface. Surprisingly, nobody considered that formulation before. Conjecture 2 holds for no orientable surface, since there exist projective graphs (hence having planar covers) of arbitrarily high orientable genera [1]. However, for nonorientable surfaces we conjecture:

Conjecture 3. *A connected graph embeds in the Klein bottle if and only if it has a Klein cover.*

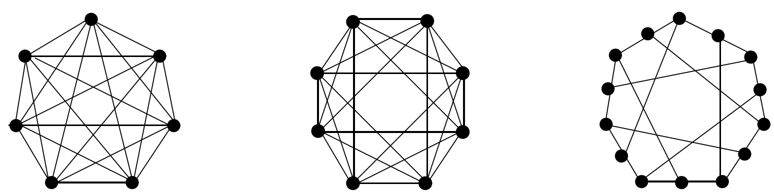

Fig. 3. The graphs $\boldsymbol{K}_7$, $\boldsymbol{K}_8 - \boldsymbol{M}_4$, and $\boldsymbol{H}_{14}$.

As far as we know, it is possible that Conjecture 3 holds for all nonorientable surfaces, but at the moment we have no evidence in favor of that. To provide support for the conjecture, we shall show that three minor-minimal graphs not embeddable in the Klein bottle do not have Klein covers: The complete graph $\boldsymbol{K}_7$, the graph $\boldsymbol{K}_8 - \boldsymbol{M}_4$ (a complete graph on 8 vertices minus a perfect matching), and the Heawood graph $\boldsymbol{H}_{14}$ (the only cubic graph of girth 6 on 14 vertices, also the geometrical dual of any toroidal embedding of $\boldsymbol{K}_7$). The author is indebted to H. Glover and B. Richter for providing these graphs.

Theorem 1. *The graphs $\boldsymbol{K}_7$, $\boldsymbol{K}_8 - \boldsymbol{M}_4$, and $\boldsymbol{H}_{14}$ have no Klein covers.*

Proof. **($\boldsymbol{K}_7$)** Suppose that $\boldsymbol{G}$ is a cover of $\boldsymbol{K}_7$ embedded in the Klein bottle. By Euler's formula, $\boldsymbol{G}$ is a 6-regular triangulation of the surface. We represent the covering projection as a labelling of the vertices by labels $1, 2, \ldots, 7$, where each label is connected with all six other labels. In particular, two vertices of the same label are at distance at least 3.

A *straight-ahead walk* is a walk in which each internal vertex is left through an edge opposite to the edge through which it was entered. This is well-defined, because every vertex of $\boldsymbol{G}$ has even degree. A key observation is that two vertices of the same label cannot be connected by a straight-ahead walk of length

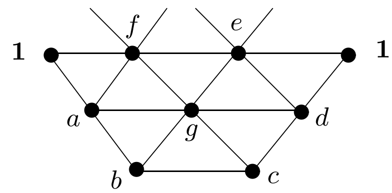

Fig. 4. A straight-ahead walk between two vertices of the same label.

three. To prove it, see Fig 4—the seven vertices a, b, c, d, e, f, g must have mutually distinct labels, but none of them may have label 1, a contradiction. (The pictures presented here should be regarded as a lifting of the graph into the universal covering surface. It could happen, for example, that in the Klein bottle embedding of $\boldsymbol{G}$ the two vertices labelled 1 are actually equal.)

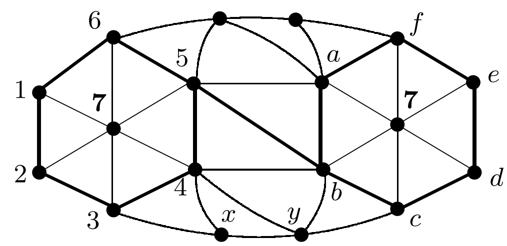

Fig. 5. A fragment of a Klein cover of $\boldsymbol{K}_7$

Let us now look at the components of the graph obtained from $\boldsymbol{G}$ by deleting all edges not contained in any closed neighbourhood of a vertex of label 7. These components are wheels with central label 7; and we will show that the rim vertices of each wheel are labelled in the same cyclic order, which contradicts the nonorientability of the Klein bottle.

We may assume, without loss of generality, that one of the wheels W is labelled $1, 2, 3, 4, 5, 6$ in order. Let another wheel labelled a, b, c, d, e, f be connected to W by at least one edge, say $b5$ (see Fig. 5). The edge $b5$ is not the only edge between these two wheels, for otherwise the central vertices would be connected by a straight-ahead walk of length three. Thus the wheels are connected by a triangle, say $ab5$. Since the vertex 5 has only one more edge, there must be another edge, say $b4$, between the wheels. Let x, y denote the other two neighbours of the vertex 4, as in the figure.

The label b cannot be $3, 4, 5, 6$ since these labels already occur at distance at most two from it; similarly a cannot be $4, 5, 6$; and c cannot be $4, 5$, and 6 since c is connected with 6 by a straight-ahead walk of length three. Then one of a, b, c is 2, so the vertex y is at distance at most two from labels $2, 3, 4, 5, 7$, and y is connected with 1 by a straight-ahead walk of length three. Hence $y = 6$, and consequently $x = 1$, which already implies $b = 2$, $c = 3$, $a = 1$. By symmetry between the two wheels, $4 = d$, $5 = e$, $6 = f$, and the claim follows by induction.

($\boldsymbol{H}_{14}$) The Heawood graph $\boldsymbol{H}_{14}$ is a bipartite cubic graph of girth 6. It is easy to check that if we replace each vertex of one colour class by a triangle on its three neighbours (a $Y\Delta$-trasformation), we get the complete graph $\boldsymbol{K}_7$. The same transformation is applicable to any cover of $\boldsymbol{H}_{14}$; thus a Klein cover of $\boldsymbol{H}_{14}$ could be transformed to a Klein cover of $\boldsymbol{K}_7$, which we have already shown is impossible.

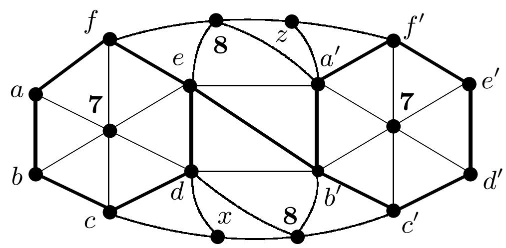

Fig. 6. A fragment of a Klein cover of $\boldsymbol{K}_8 - \boldsymbol{M}_4$

($\boldsymbol{K}_8 - \boldsymbol{M}_4$) Finally, we look at the case of $\boldsymbol{K}_8 - \boldsymbol{M}_4$. The idea of a proof is quite similar to that of $\boldsymbol{K}_7$, so we only sketch it here. Let the vertices be $1, 2, \ldots, 8$, so that the four missing edges are $12, 34, 56, 78$. Again, two vertices of the same label cannot be connected by a straight-ahead walk of length 3, for otherwise (see Fig. 4), the vertices a, b, c, d, e, f, g would get seven distinct labels other than 1, so g which is connected with all of them, should be labelled 2. By mirror symmetry, another vertex labelled 2 should be at distance two from g, a contradiction.

We partition the supposed Klein cover of $\boldsymbol{K}_8 - \boldsymbol{M}_4$ into wheels centered at labels 7, and into the remaining vertices of labels 8. We argue in the same way that two neighbouring wheels are connected by at least three edges, as in Fig. 6. Then the positions of labels 8 are determined, since each vertex other than 7 must have a neighbour labelled by 8. The labels $1, \ldots, 6$ cannot be specified since they are not mutually equivalent in this case, so we denote them (in some order) by a, b, c, d, e, f for the first wheel, and by a', b', c', d', e', f' for the second wheel. The label b' can only be a or b, and the label a' can only be a or c (there is a straight-ahead a', b-walk of length three). If $b' = a$, then $a' = c$, and the cover would contain seven edges with labels $a7, a8, ab, af, ad, ae, ac$, which is impossible. Thus $b' = b$, and we see that the cover contains edges labelled $b7, b8, ba, bc, be, bd$, so bf is one of the missing edges of the graph. Since 78 is a missing edge too, and cd, de are present in the cover, the remaining two missing edges are ce, ad. Consequently, $a' = a$ since there is an edge labelled $a'e$, and

$c' = c$; and finally, $d = d'$, $e = e'$, $f = f'$ by symmetry. ∎

Unfortunately, there is little hope to prove the conjecture by examining all forbidden minors for the Klein bottle or for higher nonorientable surfaces. Those lists are not known, and even if they are eventually found, they will probably be too numerous to be useful. However, it is worthwhile to mention that the lists are finite by [11].

3 Planar emulators

Fellows generalized Negami's conjecture to planar emulators [4] by dropping the assumption of unique neighbours. A graph $\boldsymbol{G}$ has an *emulator* $\boldsymbol{H}$ if there is a surjective mapping $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$ that maps the neighbours of each vertex v in $\boldsymbol{H}$ onto the neighbours of $\varphi(v)$ in $\boldsymbol{G}$.

Conjecture 4. (M. Fellows, 1988) *A graph has a planar emulator if and only if it has a planar cover.*

Fellows [4] gave partial evidence for the conjecture, but much less is known about planar emulators than about planar covers. The same problem, under the name of a *branched cover*, was also considered later by Kitakubo [7], who proved the equivalence for regular covers. (A cover or a branched cover $\boldsymbol{H} \to \boldsymbol{G}$ is said to be regular if $\boldsymbol{G}$ can be obtained as the quotient of $\boldsymbol{H}$ by a group action.)

One might speculate that Conjecture 4 could hold for other surfaces as well. Unfortunately it does not, as we now show.

Proposition 2. *There exists a connected graph $\boldsymbol{G}$ that has an emulator embeddable in the triple-torus, but $\boldsymbol{G}$ has no finite cover (and hence no embedding) there.*

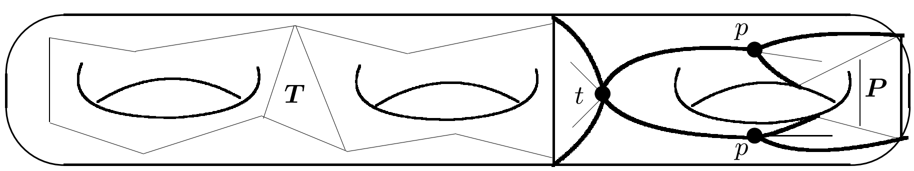

Fig. 7. An emulator of the graph $\boldsymbol{G}$.

Proof. Let $\boldsymbol{T}$ be any triangulation of the double-torus, and let $\boldsymbol{P}$ be any projective graph that does not embed in the triple-torus [1]. We construct a graph $\boldsymbol{G}$ by connecting some vertex t of $\boldsymbol{T}$ with some vertex p of $\boldsymbol{P}$. This graph obviously does not embed in the triple-torus. A possible emulator is obtained by drawing $\boldsymbol{T}$ on two of the handles of the triple-torus, putting a double-cover of $\boldsymbol{P}$ on the third handle, and connecting the vertex t with both of the vertices covering p, as shown in Fig. 7.

Suppose that there is a cover of the graph $\boldsymbol{G}$ that embeds in the triple-torus. Euler's formula applied to $\boldsymbol{T}$ shows that it may be at most a double-cover, and then the subcover of $\boldsymbol{T}$ triangulates the whole triple-torus. But in such case, it is impossible to connect both of the vertices covering p with the subcover of $\boldsymbol{P}$, a contradiction. ∎

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