Stars and Bonds in Crossing-Critical Graphs

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Abstract

The structure of all known infinite families of crossing–critical graphs has led to the conjecture that crossing–critical graphs have bounded bandwidth. If true, this would imply that crossing–critical graphs have bounded degree, that is, that they cannot contain subdivisions of $K_{1,n}$ for arbitrarily large n. In this paper we prove two results that revolve around this conjecture. On the positive side, we show that crossing–critical graphs cannot contain subdivisions of $K_{2,n}$ for arbitrarily large n. On the negative side, we show that there are graphs with arbitrarily large maximum degree that are 2-crossing–critical in the projective plane.

Keywords: crossing number, crossing-critical graph, bounded degree, bandwidth

1 Crossing Numbers and Crossing–Critical Graphs

The *crossing number* $\mathrm{cr}_\Sigma(G)$ of a graph G in a surface Σ the minimum number of pairwise crossings of edges in a drawing of G in Σ. Whenever the reference to Σ is omitted, it is assumed that Σ is the plane (or, equivalently in the realm of crossing numbers, the sphere).

Calculating the exact crossing number of a graph is a computationally hard problem, and for many years most crossing number papers focused on calculating or estimating the crossing number of interesting families of graphs. This trend has been reversed in the last few years, as questions of a more structural character have been successfully tackled (see for instance [2,5,9]).

As with other classical graph theoretical parameters, we gain a great insight into crossing numbers by looking at graphs that are minimal with respect to having a certain crossing number. A graph G is *k–critical* in Σ if $\mathrm{cr}_\Sigma(G) \geq k$ but $\mathrm{cr}_\Sigma(G - e) < k$ for each edge e of G. Many interesting questions and results in crossing–critical graphs in the plane are related to the work of Richter and Thomassen [9].

A little over two decades ago, Širáň [11] and Kochol [7] gave nice constructions of crossing–critical graphs. Kochol's family of critical graphs has inspired a good deal of research. These constructions have been generalized by several authors [10,1]. All such generalizations share one key feature from Kochol's original construction; the infinite families consist of "long and thin" graphs. This led Salazar and Thomas to conjecture that crossing–critical graphs have bounded path–width (see [3]). This conjecture has been proved in [5].

Thomassen observed that all constructions known, including the slightly different flavoured constructions by Hliněný [4,6], satisfy the stronger property of having bounded bandwidth. A graph G has *bandwidth* at most k if there is a bijection $\beta : V(G) \to \{1, \ldots, |V(G)|\}$ such that $|\beta(u) - \beta(v)| \leq k$ for each edge $e = uv$ in G. This observation has been recorded as a (still open) conjecture by Richter and Salazar in [8].

Conjecture 1.1 *For each integer $k > 0$ there is a number $B(k)$ such that if G is k–crossing–critical, then the bandwidth of G is at most $B(k)$.*

The following weaker form of Conjecture 1.1 remains also open:

Conjecture 1.2 *For each integer $k > 0$ there is a number $D(k)$ such that if G is k–crossing–critical, then the maximum degree of G is at most $D(k)$.*

In this paper we present two results inspired by Conjecture 1.2. In the direction supporting this conjecture, we show the following.

Theorem 1.3 *For each integer $k > 0$, there is a $f(k)$ such that if G is k–crossing–critical, then G does not contain a subdivision of $K_{2,f(k)}$. In particular, $f(k) \leq 30k^2 + 200k$.*

Supporting the viewpoint that Conjecture 1.2 is false, we show that its projective plane version (and consequently the projective plane version of Conjecture 1.1) is false.

Theorem 1.4 *There is an infinite family of simple 3-connected graphs H_k, $k \geq 3$, such that each H_k is 2-crossing-critical in the projective plane and has a vertex of degree $6k$.*

2 Bridges in crossing–critical graphs: Theorem 1.3

In order to show that no large $K_{2,n}$ subdivisions exist in a k–crossing–critical graph (k is fixed), we first take any two vertices u, v, thinking of them as the degree-n vertices in a $K_{2,n}$ subdivision in G. We wish to analyze the $\{u, v\}$–bridges in G. We use "bridge" in the sense of Tutte: a $\{u, v\}$–*bridge* in G is either a single edge with endpoints u and v, including u and v (a *trivial* bridge), or a subgraph of G obtained by adding to a component H of $G - u - v$ the vertices u and v and all edges attaching H to u and v.

Our first aim in this section is to prove:

Claim 2.1 *If u, v are the degree-n vertices of a large $K_{2,n}$ subdivision in G, then a large number of u–v paths are drawn (in every optimal drawing of G) inside a closed disc Δ bounded by two u–v paths, in such a way that the chunk of G drawn in Δ is crossing-free and connected even after the removal of u, v.*

This is a central prerequisite in the proof of Theorem 1.3.

For that we need two preliminary results. First we study the implications of a large enough number of $\{u, v\}$–bridges: in an optimal drawing of such a (not necessarily critical) graph, distinct $\{u, v\}$–bridges are disjointly drawn, and a single face of the drawing is incident with both u and v. Then we show that the number of $\{u, v\}$–bridges is bounded in k. Finally, taking this bound and the total number of crossings in G into an account, we establish Claim 2.1.

Then we move onto proving Theorem 1.3 itself: Seeking a contradiction, we suppose that G contains a large $K_{2,n}$ subdivision, i.e. at least n pairwise internally disjoint $\{u, v\}$–paths, and G is k–crossing-critical at the same time. It is easily seen that G may be assumed 2-connected.

We pick $\mathcal{D}$ an optimal drawing of G. We focus on the (unique) $\{u, v\}$–bridge F of G that contains the chunk of G drawn under Δ, as established in

Claim 2.1. We pick an edge e of F incident with u in the "middle of Δ", and examine an optimal drawing $\mathcal{D}_e$ of $G - e$ which has less than $\mathrm{cr}(G)$ crossings by the criticality assumption. To prove Theorem 1.3 it suffices to show:

Claim 2.2 *There exists a drawing $\mathcal{F}$ of F which: (a) has no more crossings that $F - e$ has in $\mathcal{D}$, and (b) the "face distance" between u and v in $\mathcal{F}$ is not larger than in $\mathcal{D}(F)$.*

Indeed, having such $\mathcal{F}$ at hand, we simply replace the subdrawing of $F - e$ in $\mathcal{D}_e$ with $\mathcal{F}$. This yields a drawing $\mathcal{D}'$ of G again. By (a), there are no more crossings involving two edges of F in $\mathcal{D}'$ than previously, and by (b) the replacement operation may be performed so that there are no more crossings involving one edge of F and one edge of other $\{u, v\}$–bridge of G. Hence $\mathrm{cr}(G - e) \geq \mathrm{cr}(G)$, providing the required contradiction.

To produce the drawing $\mathcal{F}$ of F from Claim 2.2, we combine chunks of the drawings of F in $\mathcal{D}$ and in $\mathcal{D}_e$. Here we heavily use the fact that we have many crossing-free $\{u, v\}$–paths in F both under $\mathcal{D}$ and $\mathcal{D}_e$.

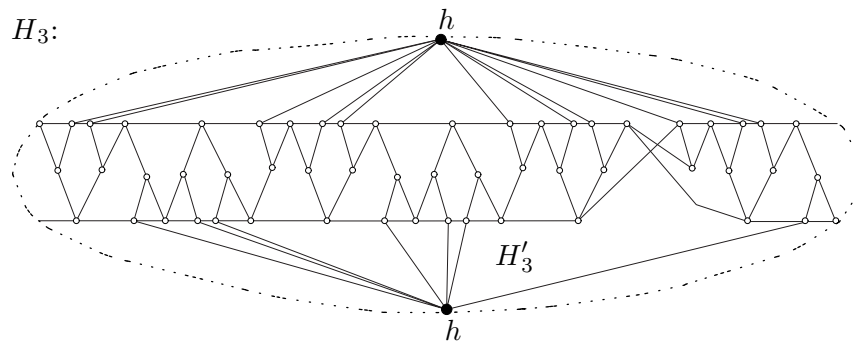## 3 Projective critical graphs: Theorem 1.4

Fig. 1. The graph H_3 drawn in the projective plane with two crossings (the pairs of opposite points on the dashed ellipse get identified).

The aim of the last section is to provide the construction of a graph family H_k, $k = 3, 4, \ldots$, claimed in Theorem 1.4. Here we turn the idea of a classical "twisted" critical construction inside out to produce an untwisted planar belt which we consequently force to "twist" in the projective plane by adding an additional high-degree vertex, connected to both sides of the belt.

We refer the reader to Fig. 1 for a taste of our construction. This is a projective drawing of H_3, obtained from three copies of a certain tile. Its extension to higher values of k is quite straightforward.

We rather easily show that the crossing number of H_k minus any edge is at most 1. Then comes the hard part—proving that the projective crossing number of H_k is at least 2. The main tool for this last part is a structural analysis: We examine each edge e of H_k and show that, for most choices of e, the subgraph $H_k - e$ still contains one of the forbidden minors for the projective plane, and hence the crossing number must be $\geq 1 + 1$. For the remaining few edges of H_k, we show that if any two of them cross each other, then the resulting drawing must contain yet another crossing. Again, the structural tool is in establishing presence of forbidden minors for the projective plane.

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