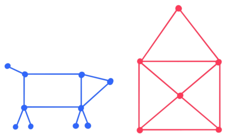
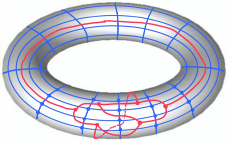
On Hardness of the Joint Crossing Number





Petr Hliněný *

Faculty of Informatics, Masaryk University
Brno, Czech Republic

joint work with **Gelasio Salazar**

Instituto de Fisica, Universidad Autonoma
de San Luis Potosi, Mexico

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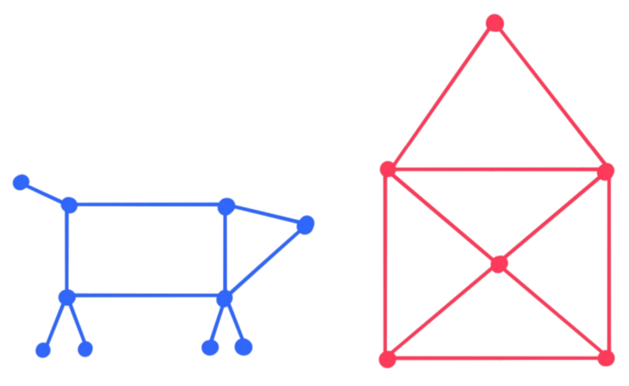

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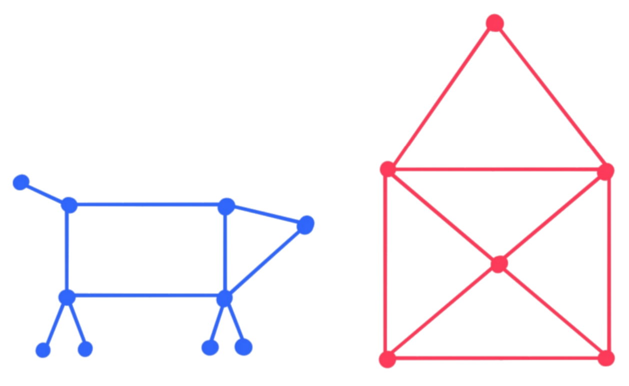

- Silly question → of course, no crossings are needed!

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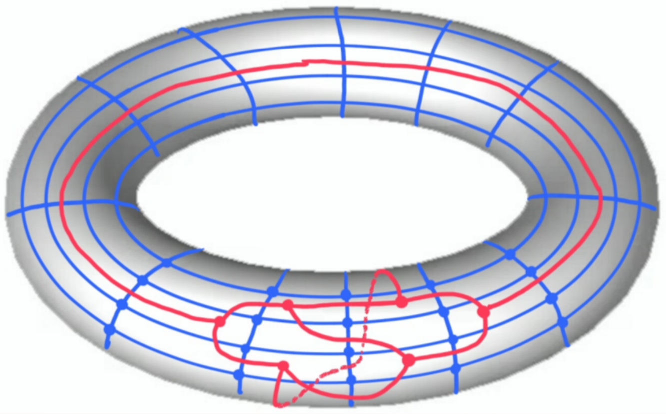

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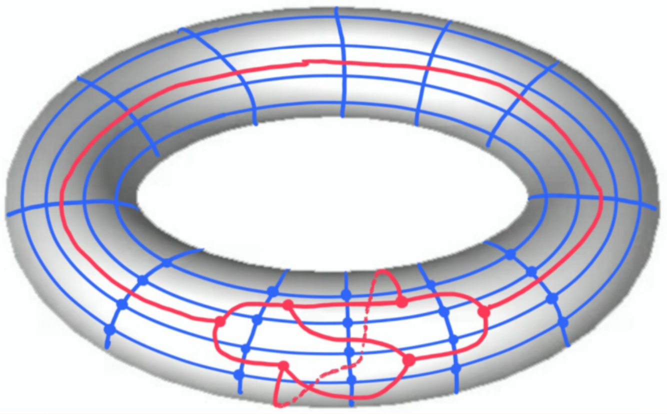
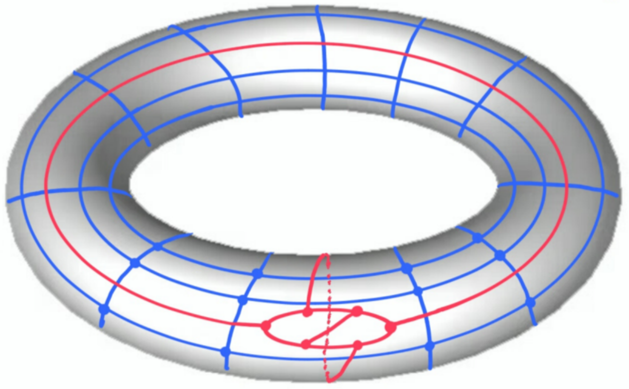


→ Hence, indeed, some mutual crossings are needed even if each one of the two graphs (itself) embeds there.

Two Graphs in one Surface

An easy solution?

- Actually, why should these two graphs be entwined on the torus?
  We can perhaps do better using just one (good) face of each map...



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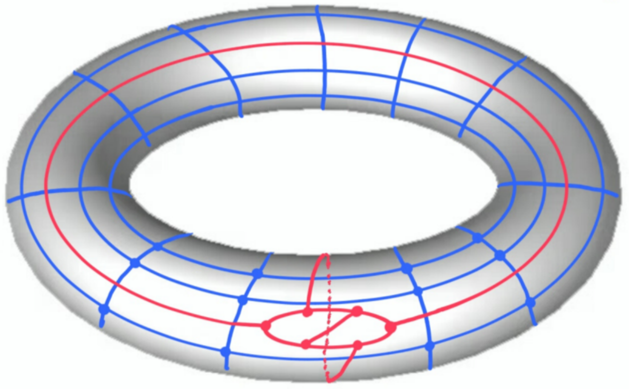

- NO; this tempting toroidal example is very misleading!

2 Joint Embedding: a Brief History

To minimize the number of mutual edge crossings in a joint embedding of two graphs (say, red and blue) in one common surface.

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- [Richter–Salazar, 2005]: Disproving the A.–B. conjecture in the double-torus, a replacement conjecture given.
  Improved Negami's upper bound wrt. representativity.
- And more. . . ?

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    and so it makes better sense to prove hardness without assuming artificial restrictions, but make the construction working with all the restrictions (e.g., homeomorphism).

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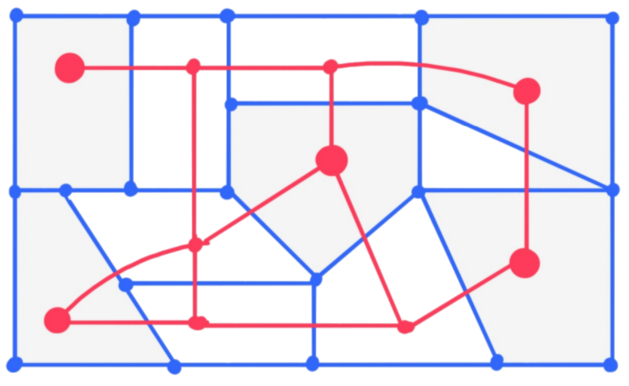

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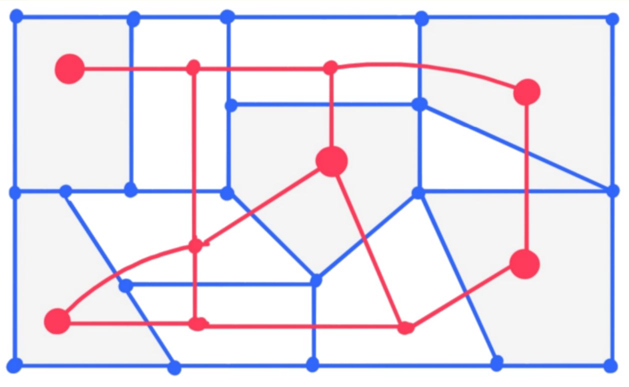

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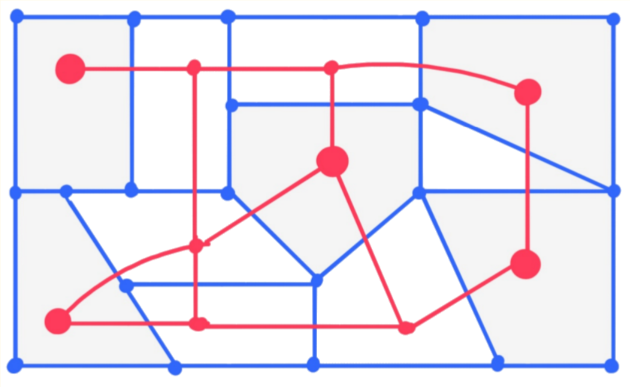

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- Need to show that *face-anchors* can be enforced in a joint embedding...

Highly Entwined Drawings, I

Getting to the plane

How?

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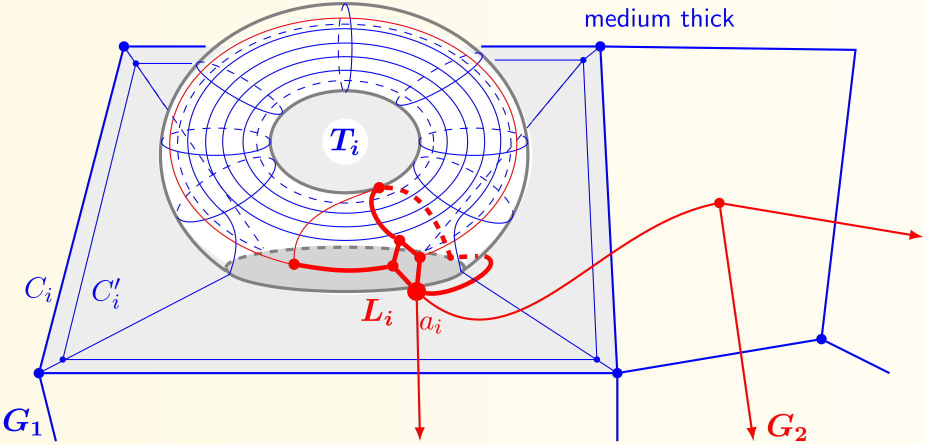
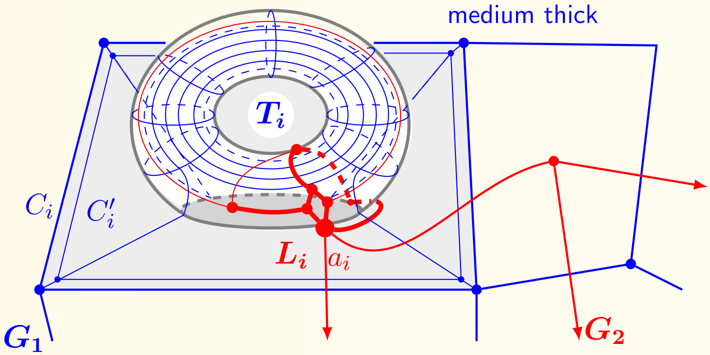
Getting to the plane

How? Use the following *gadget* for each face-anchor (the anchor is thick red):



The gadget and the construction



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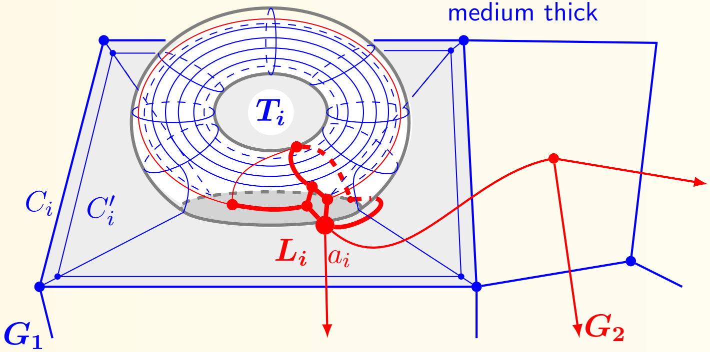

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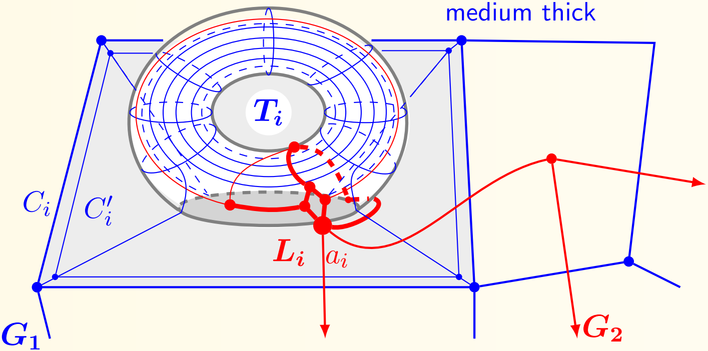

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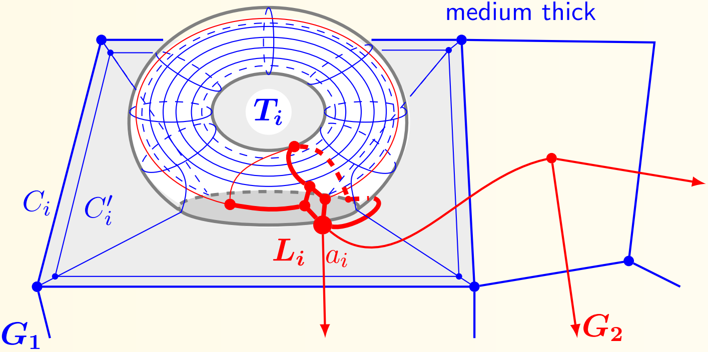

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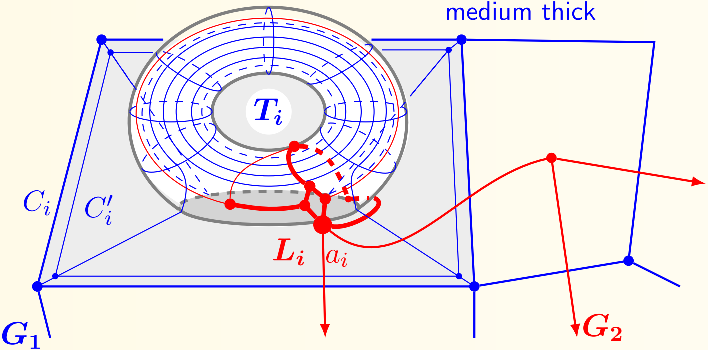

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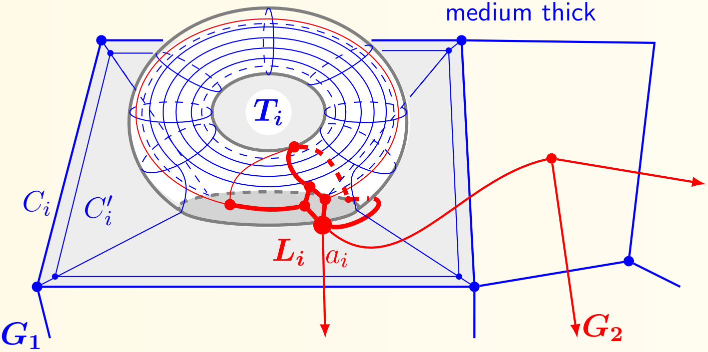

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- Playing slightly with the weights of the red $K_{3,3}$ s and the blue grids, we can enforce a precise *one-to-one assignment* (and no other permutations).

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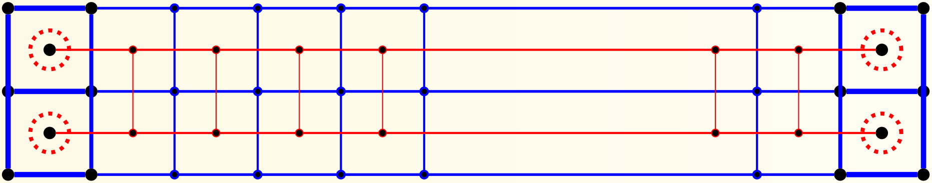
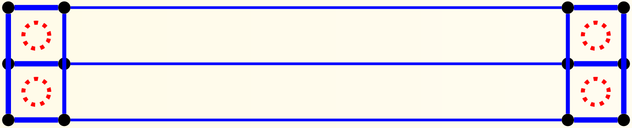
The high level idea of anchor multiplication – a **multi-anchor gadget**:



- Only *four* face-anchors are used to tie down two long vertex sequences.

The multi-anchor gadget

1. Make the base blue frame very thick to thick:



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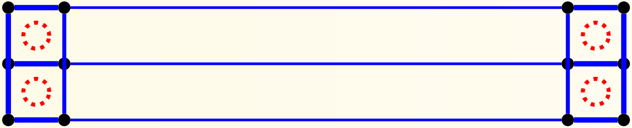
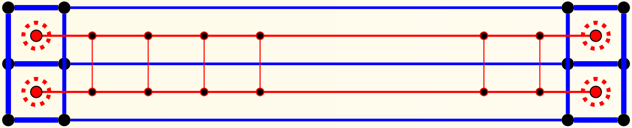


2. Stretch the thinner red ladder through that frame – enforced this way:



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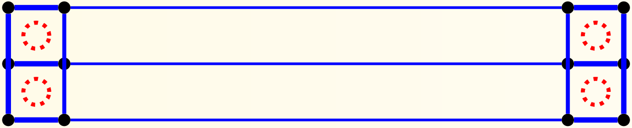

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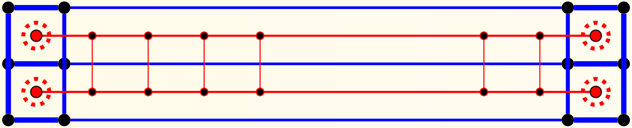
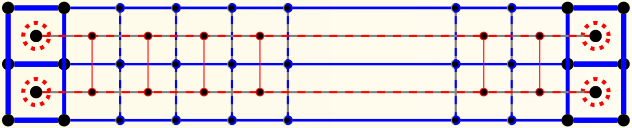
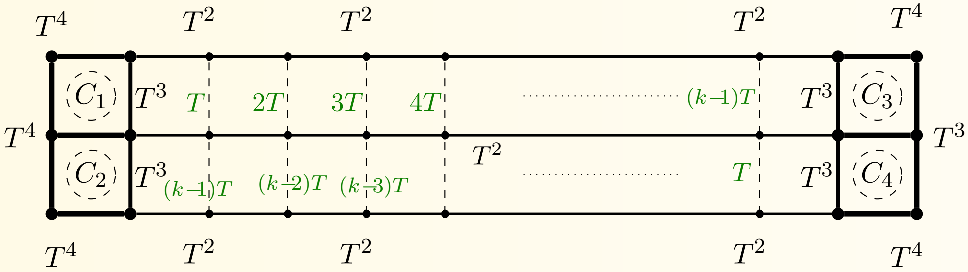


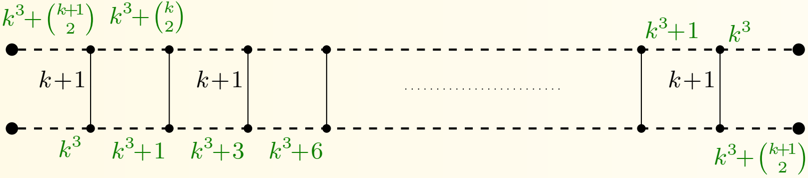
3. Adjust weights on the horizontal red and on new (med.-light) vertical blue bars to enforce unique even distribution of the red ladder vertices.



Gadget details

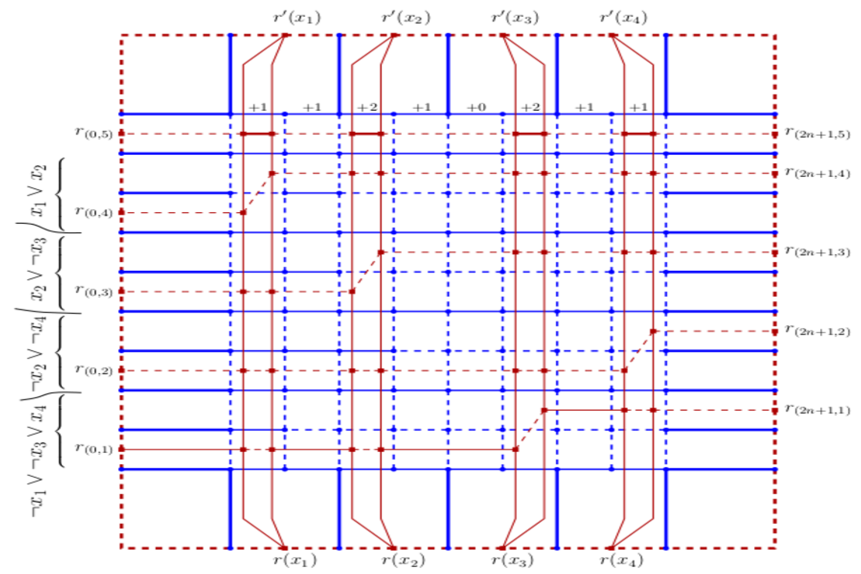
How thick the edges are? $T >> k >> 1$





5 Anchored Hardness Reduction

[Cabello–Mohar] (2012): *Anchored planar joint crossing number* is NP-hard:



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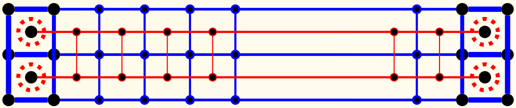

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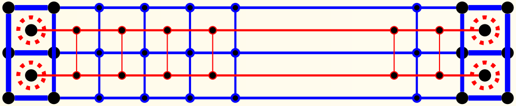

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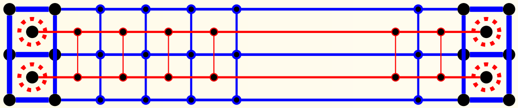

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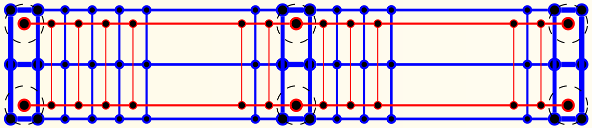
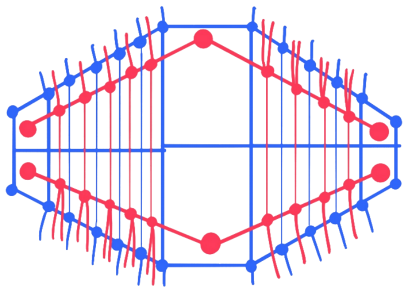
- Two copies of the gadget to emulate the four sides of the C.–M. constr.:



Putting all together

Double multi-anchor



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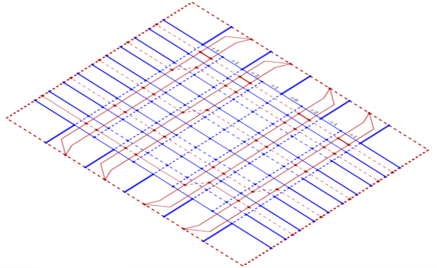
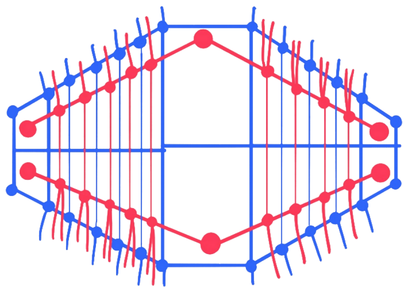
Cabello–Mohar



Putting all together

Double multi-anchor



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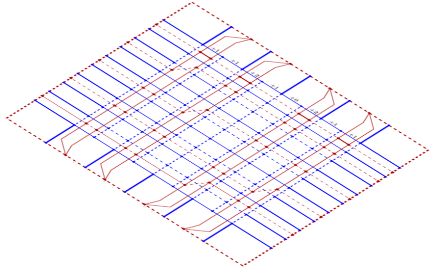

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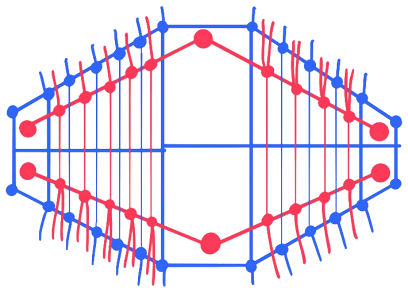

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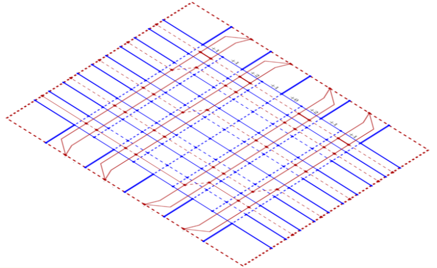

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Theorem. JOINT CROSSING NUMBER, JOINT HOMEOMORPHIC CROSSING NUMBER, and JOINT OP-HOMEOMORPHIC CROSSING NUMBER are NP-hard problems in any orientable surface of genus 6 or higher. This remains true even if the inputs are restricted to simple 3-connected graphs.

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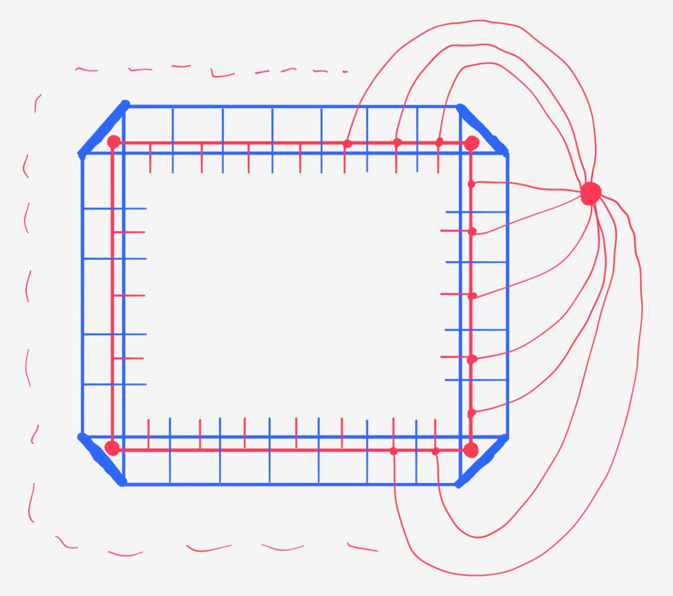
(Previously, [Cabello–Mohar] required an unlimited number of degrees > 3.)

The improved multi-anchor gadget

Just a simple sketch. . .



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Thank you for your attention.