Faster Existential FO Model Checking on Posets

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    ↑ MSO_1 vs. MSO_2 ↓
  - or, quantifies vertex and edge sets together $\exists X, Y, E, F$.

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  - [Grohe–Kreutzer–Siebertz] **nowhere dense graphs !** (2013)

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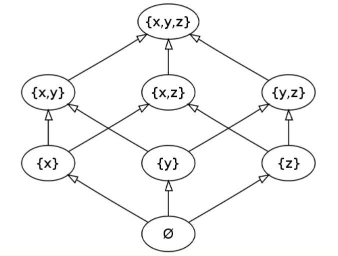

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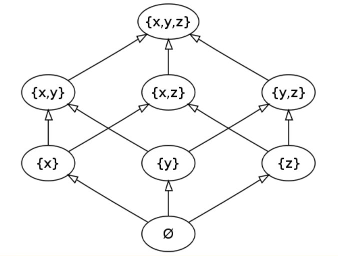

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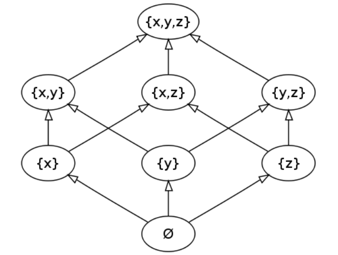

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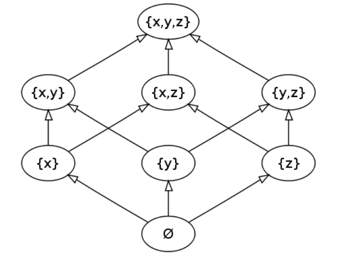

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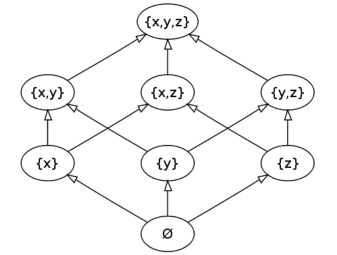

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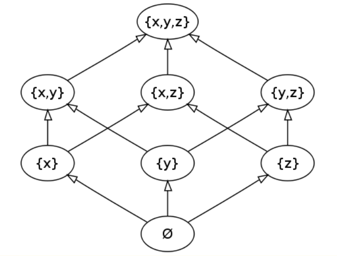

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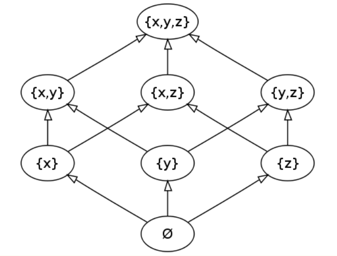

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while the problem **becomes FPT** for

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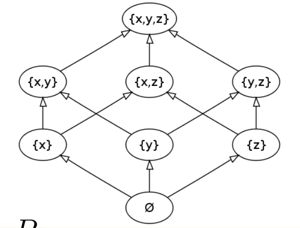
A short detour; Hasse Diagrams



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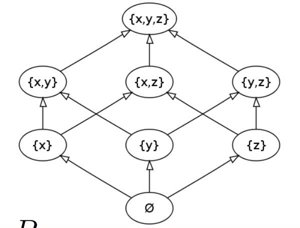

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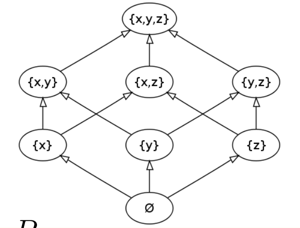

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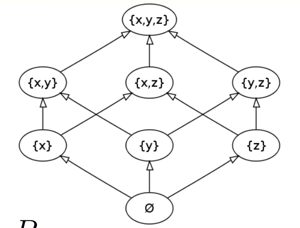

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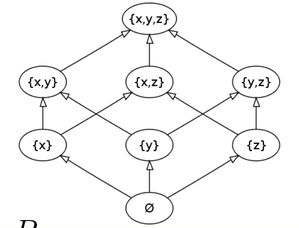

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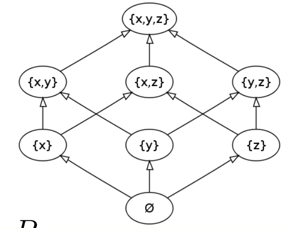

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- Step 3. Solving the homomorphism problem in polytime (using a highly non-trivial theorem).

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  - an injective mapping $e : Q \to P$ such that
    $q \leq_Q q'$ iff $e(q) \leq_P e(q')$, for every $q, q' \in Q$.

- Technically easier (with $width(\mathcal{P})^{|Q|}$ blow-up) to consider:

  COMPATIBLE EMBEDDING

  ... $\mathcal{P}$ with a chain partition $(C_1, \ldots, C_w)$, $w = width(\mathcal{P})$,
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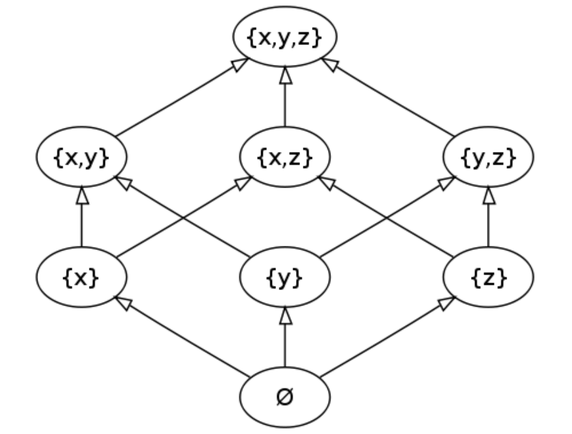
The Embedding Problem; examples

Does it embed?



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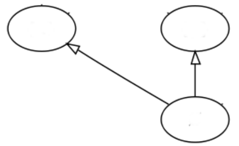

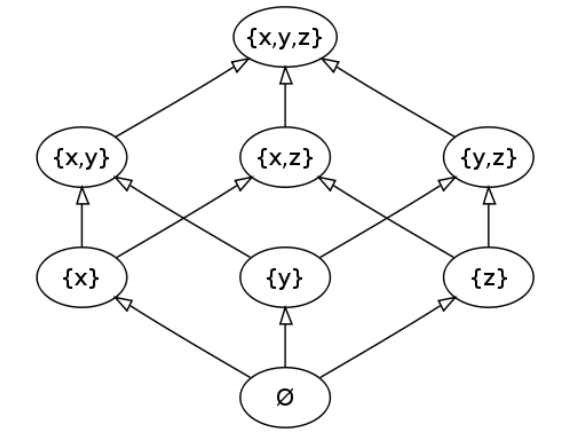

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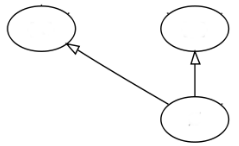

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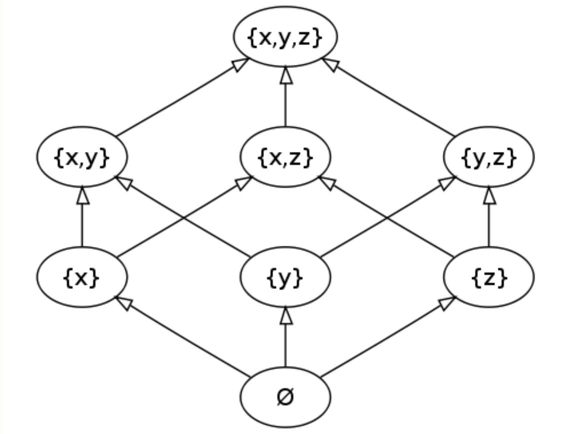

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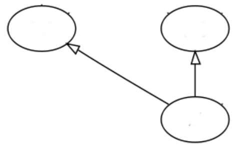

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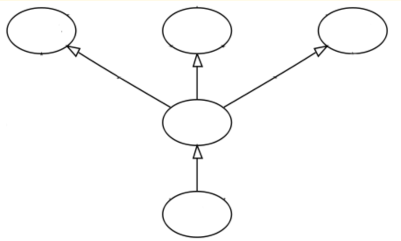

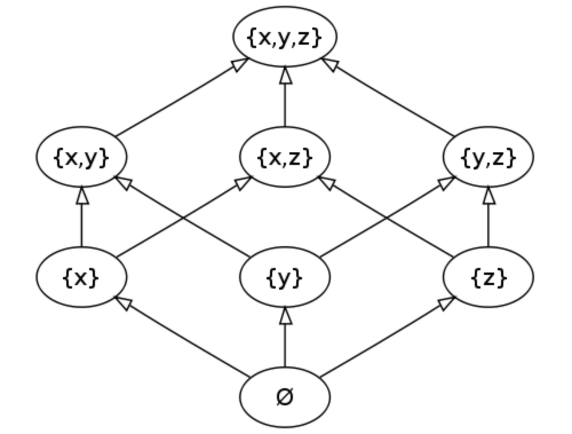

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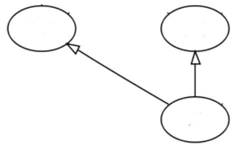

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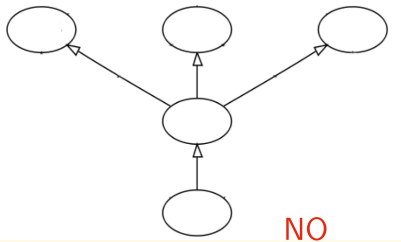

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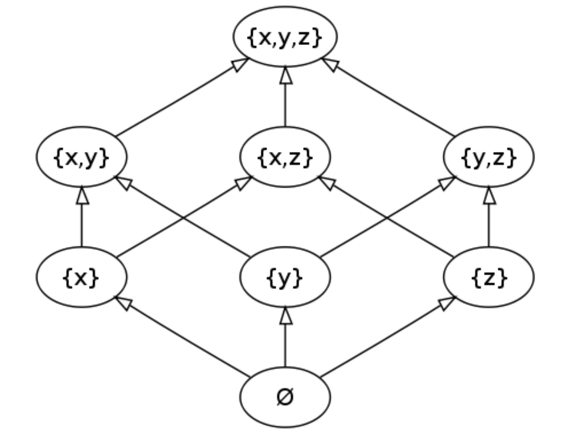

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$$\boxed{h(|\phi|, width(\mathcal{P})) \cdot |P|^2.}$$

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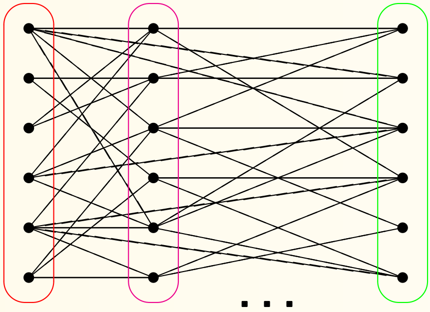
- Step 3. Solving the (interval-monotone) variant of MULTICOLOURED CLIQUE in polynomial time – overall quadratic in $|P|$.

Multicoloured Clique



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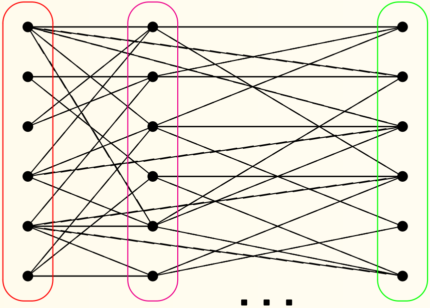

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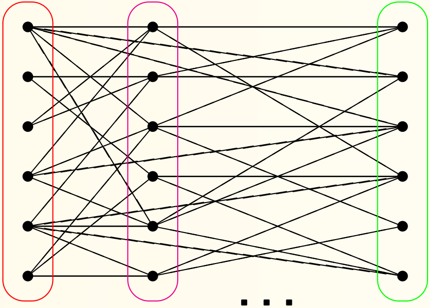

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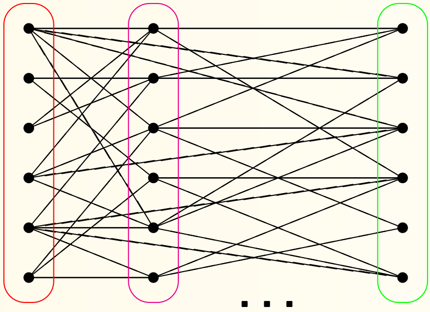

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Reduction to Multicol. Clique

- Recall... COMPATIBLE EMBEDDING
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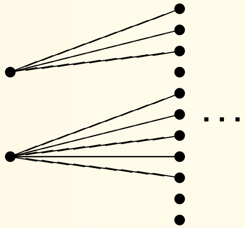
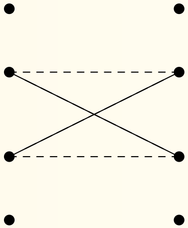
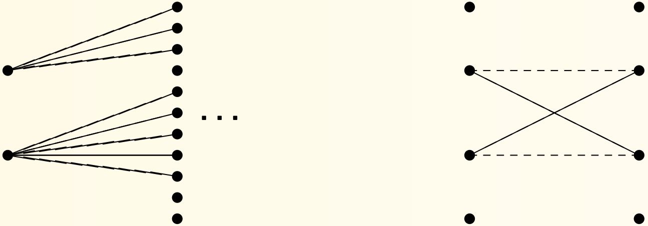
  - $V(G) = V_1 \,\dot\cup \ldots \dot\cup\, V_{|Q|}$ where V_i is a copy of $C_{f(i)}$,
  - for $p \in V_a, q \in V_b$ copies of $p' \in C_{f(a)}, q' \in C_{f(b)}$, $a \neq b$, $pq \in E(G)$ iff $p' \leq_P q' \leftrightarrow a \leq_Q b$ and $p' \geq_P q' \leftrightarrow a \geq_Q b$.





Interval-Monotone Multicol. Clique

Using special properties of the reduction...



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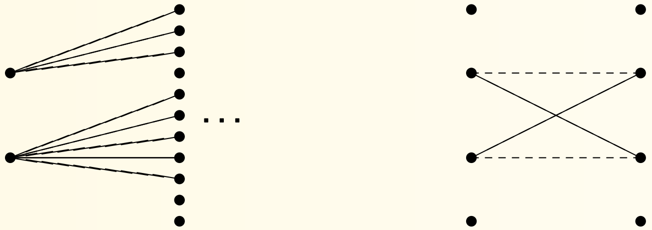

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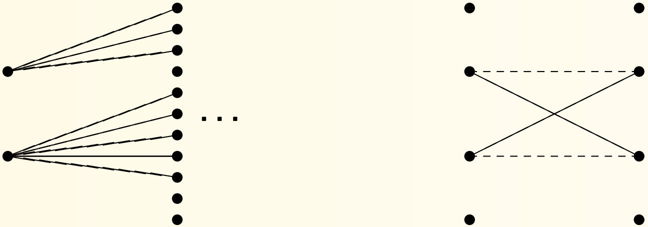

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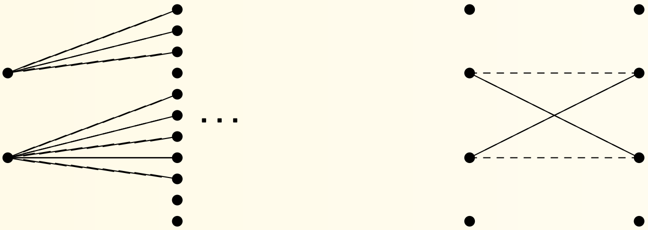

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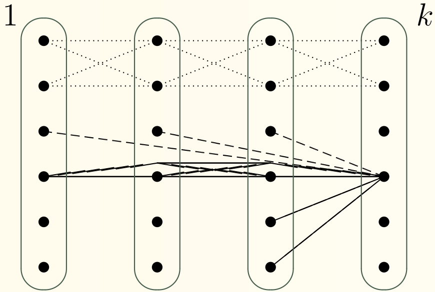
The Algorithm



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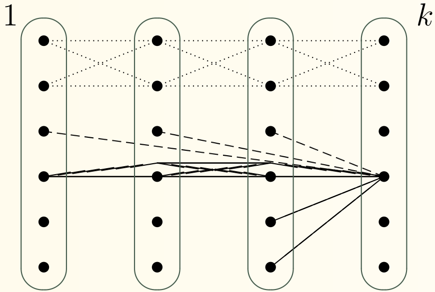

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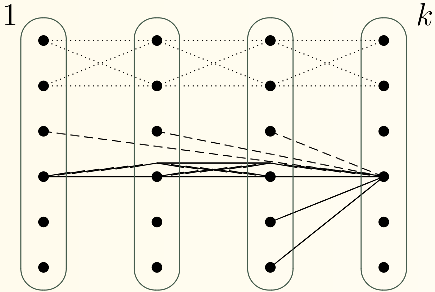

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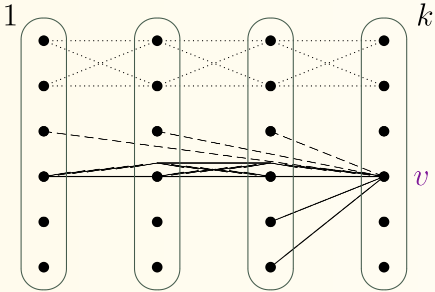

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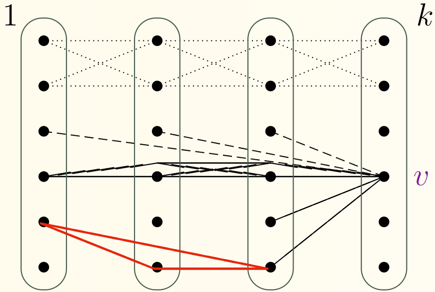

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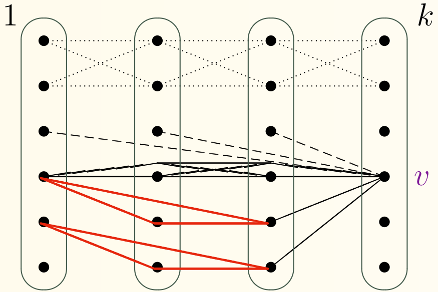

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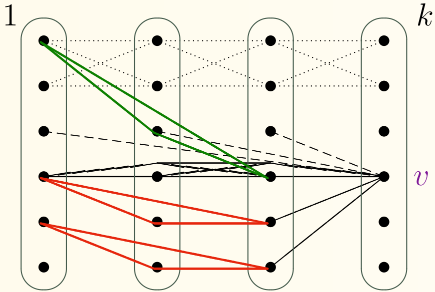

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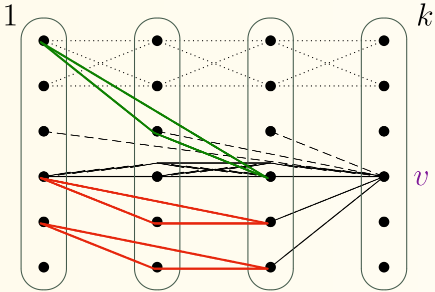

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- Improvement over [Bova, Ganian, Szeider, LICS 2014]

[BGS14]	$O(f(\phi) \cdot n^{g(w)})$
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