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Crossing Number is Hard for Cubic Graphs

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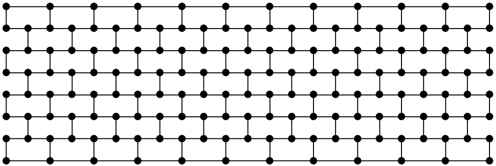
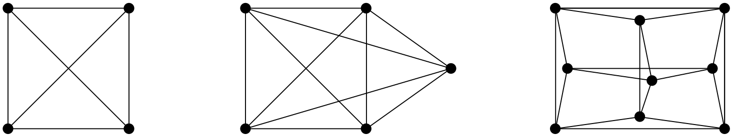


2000 Math Subjects Classification: 05C10, 05C62, 68R10

1 Drawings and the crossing number

Definition. *Drawing of a graph* G:

- The vertices of G are distinct points,
  and every edge $e = uv \in E(G)$ is a simple curve joining u to v.
- No edge passes through another vertex,
  and no three edges intersect in a common point.



Definition. *Crossing number* $\mathrm{cr}(G)$
is the smallest number of edge crossings in a drawing of G.

Origin – Turán's work in brick factory, WW II.

Importance – in graph visualization, VLSI design, etc.

Warning. There are variations of the definition of crossing number, not yet proved to be equivalent. (Like counting crossing or odd-crossing pairs of edges.)

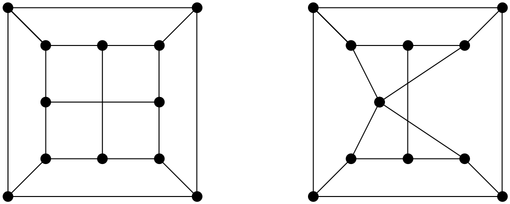
Different versions:

- *Rectilinear* crossing number – requires edges as straight lines.
  Same up to $\mathrm{cr}(G) = 3$, then much different.

- *Minor-monotone* crossing number – closes the definition down to minors.
  (Usual crossing number may grow up rapidly when contracting an edge!)

Definition. *Minor-monotone crossing number*

$$\mathrm{mcr}(G) = \min_{H:\, G \leq H \text{ (minor)}} \mathrm{cr}(H)\,.$$



Observation. (Fellows) If a cubic graph F is a minor of G, then F is in G as a subdivision. Hence for cubic F

$$\mathrm{cr}(F) = \mathrm{mcr}(F)\,.$$

2 Computational Complexity

Remark. It is (practically) very hard to determine crossing number.

Observation. The problem $\textsc{CrossingNumber}(\leq k)$ is in NP:
Guess a suitable drawing of G, then replace crossings with new vertices, and test planarity.

. . . . . .

Theorem 2.1. (Garey and Johnson, 1983) $\textsc{CrossingNumber}$ *is NP-hard.*

Theorem 2.2. (Grohe, 2001) $\textsc{CrossingNumber}(\leq k)$ *is in FPT with parameter k, i.e. solvable in time $O(f(k) \cdot n^2)$.*

. . . . . .

Our results:

Theorem 2.3. $\textsc{CrossingNumber}$ *is NP-hard on simple cubic graphs.*

Corollary 2.4. *The minor-monotone version is also NP-hard.*

NP-reduction from OLA

OptimalLinearArrangement:

Input: An n-vertex graph G, and an integer a.

Question: Is there a bijection $\alpha : V(G) \to \{1, \ldots, n\}$ (a *linear arrangement* of vertices) such that the weight of α is

$$\sum_{uv \in E(G)} |\alpha(u) - \alpha(v)| \;\leq\; a\,? \tag{1}$$

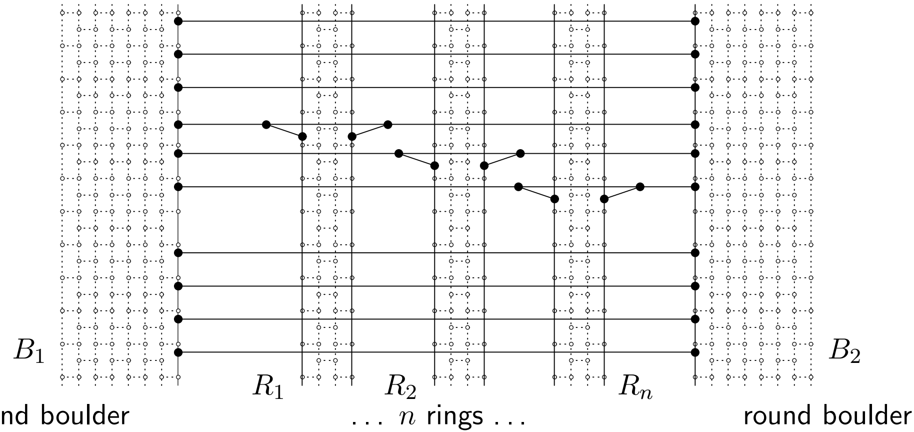
- NP-complete by [Garey and Johnson].
- Used to prove Theorem 2.1.
- Used (differently) also to prove our Theorem 2.3.

Question: What about a reduction from Planar 3-SAT?

– Tried (reasonably hard) several times, but yet unsuccessful. . . Why?

3 Sketch of a Proof

For an instance G, a of OLA, construct a graph H_G, and test $\mathrm{cr}(H_G)\dots$

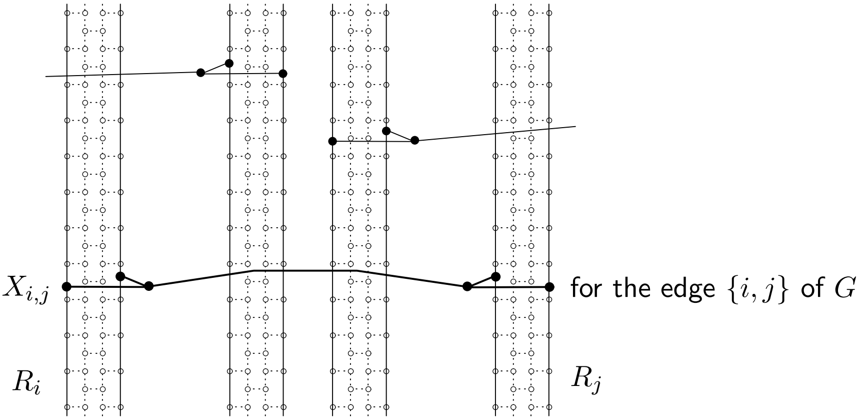


Boulders – huge, and keeping everything “in place”.

Rings – one for each vertex of G, their order is (like) α.

Spokes – horizontal connections, force rings to lie “between” the boulders.

Handles for the edges of G are attached to the (above) skeleton as follows:



Lemma 3.1. *Let us, for a given graph G on n vertices and m edges, construct the graph H_G as described above. If G has a linear arrangement of weight A, then*

$$\mathrm{cr}(H_G) \le (s+rn)nt + 2(A+m)t - 4m\,,$$

where $s+rn$ is the total number of spokes, and t is the thickness of each ring.

Converse direction

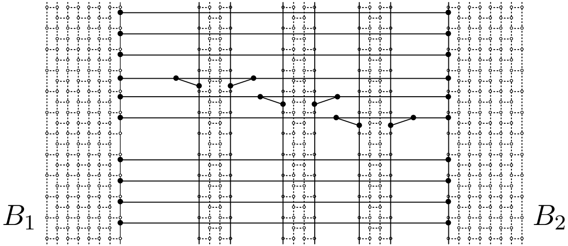
The following statement, together with Lemma 3.1, validates our reduction.

Proposition 3.2. *If an optimal linear arrangement of G has weight A, then*

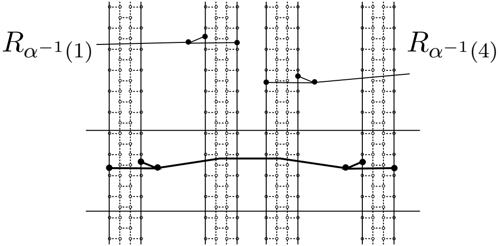
$$\mathrm{cr}(H_G) \geq (s + rn)nt + 2(A + m)t - 8m\,.$$

We proceed the proof along the following sequence of claims:

- In the optimal drawing of H_G, the boulders B_1, B_2 are drawn with no edge crossings.
- In the optimal drawing of H_G, each main cycle of every ring R_i is drawn as a closed curve separating the subdrawing of B_1 from that of B_2.



- The rings are drawn (almost) in an order $R_{\alpha^{-1}(1)}, \ldots, R_{\alpha^{-1}(n)}$.
- Drawings of the edge handles can be separated (by curves of some spokes) into disjoint areas of the plane.



- The separated edge handles generate (almost) as many edge crossings as expected.

Hence we determine the OLA value A for G from

$$(s+rn)nt + 2(A+m)t - 4m \geq \mathrm{cr}(H_G) \geq (s+rn)nt + 2(A+m)t - 8m\,.$$