Minimizing an Uncrossed Collection of Drawings

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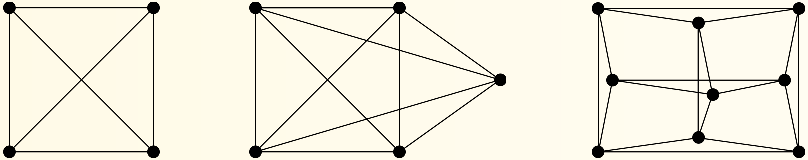
Warszawa, Poland

1 Drawings and Crossing Minimization



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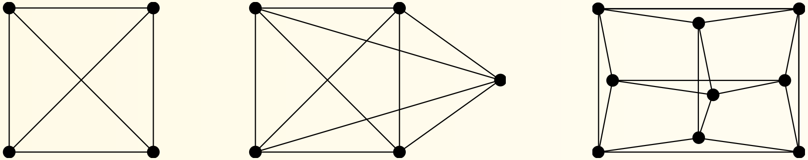

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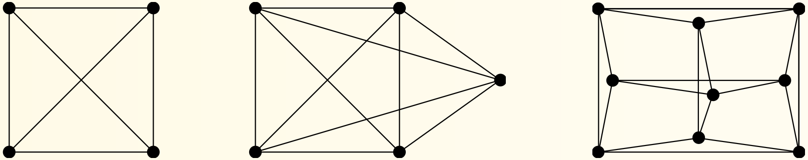

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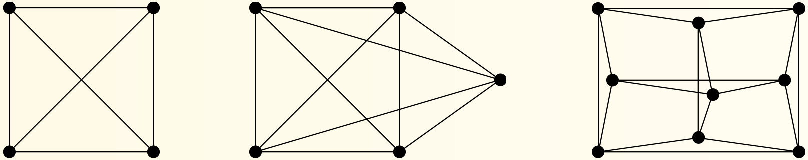

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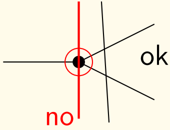

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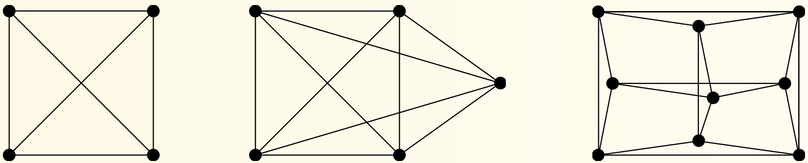

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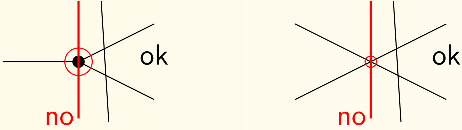

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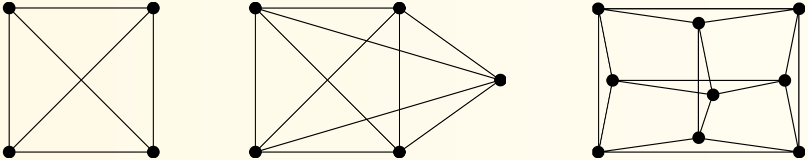

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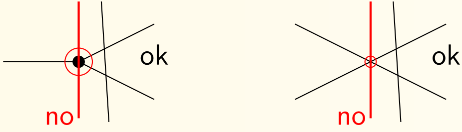
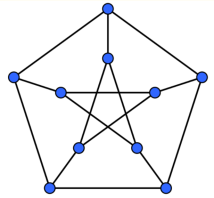
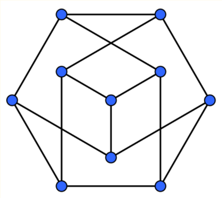
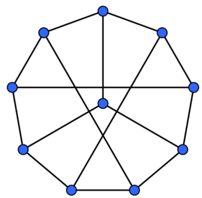


- A very hard algorithmic problem, indeed...

The Art of Multiple Drawings?







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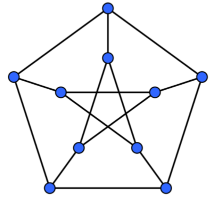

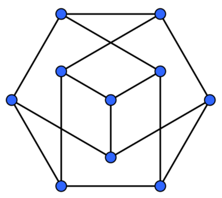

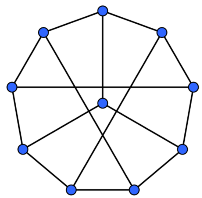

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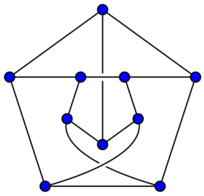

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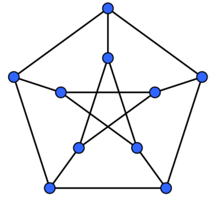

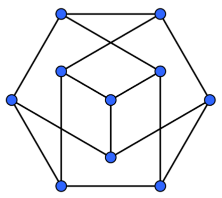

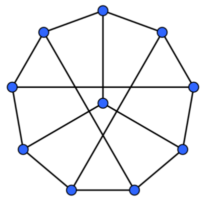

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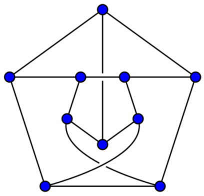

- Actually; Biedl, Marks, Ryall, and Whitesides, **GD 1998**:
  Graph Multidrawing: Finding Nice Drawings Without Defining Nice

  . . . the multidrawing approach calls for systematically producing many drawings of the same graph, where the drawings presented to the user represent a balance between aesthetics and diversity . . .

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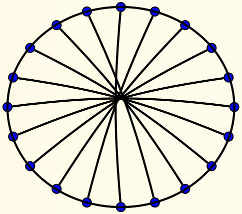
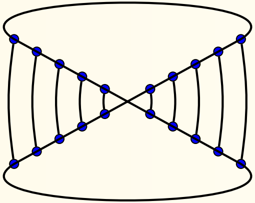
- However, what is an exactly definable view of diversity (of solutions – the drawings) for problems related to edge crossings?

Definition. A family of drawings $D_1, D_2, \ldots, D_k$ of G is an *uncrossed collection of drawings* if each edge of G is uncrossed in some D_i.





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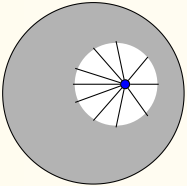
Trivial. $\mathrm{unc}(G) \leq |V(G)|$ since we can partition into stars.



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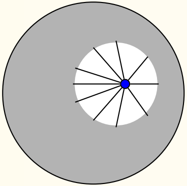

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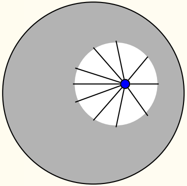

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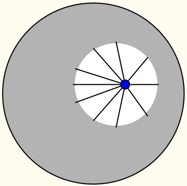

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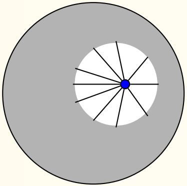

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Definition. The *(outer) thickness* of a graph G is the minimum number of *(outer) planar* subgraphs the edge set of G can be partitioned into.

Thickness, Uncrossed Number, and Outerthickness

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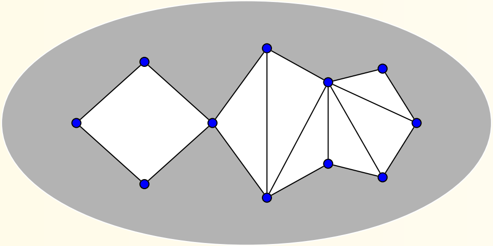

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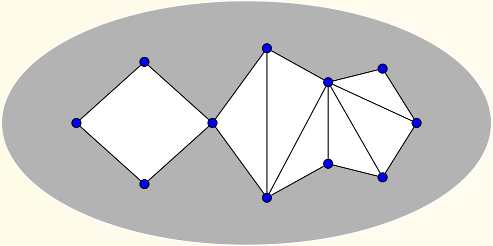

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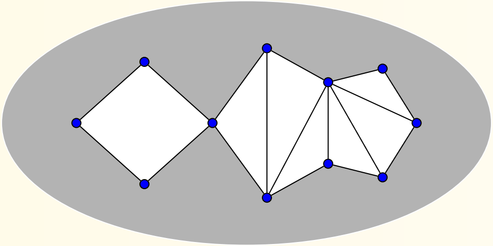

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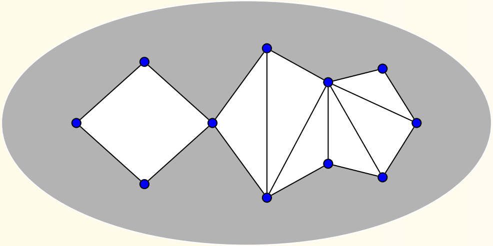

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Conjecture. Minimizing the uncrossed number is **para-NP-hard**.

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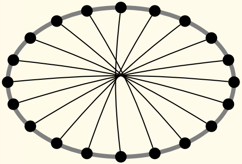

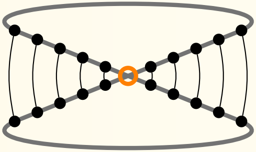

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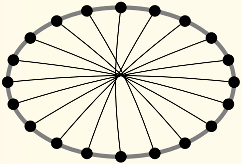

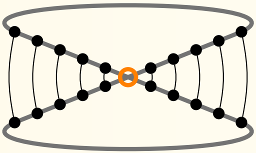

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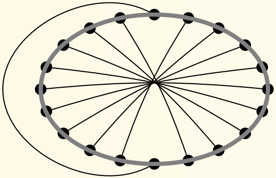

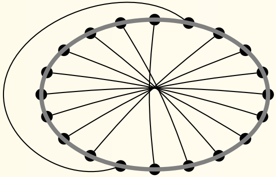

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Observation. Since $\mathrm{cr}(K_n) \in \Theta(n^4)$, and $\mathrm{unc}(K_n) \in \Theta(n)$, we have

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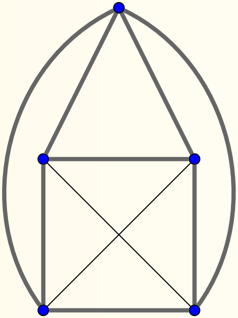
Proof. K_5 with suitable "thick" edges has $\mathrm{cr}(K_5^*) = 1$ but unbounded $\mathrm{ucr}(K_5^*)$ since any other choice of a crossing is very costly.



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Theorem (the Crossing Lemma). [Ackerman 2019]
If G is simple and $|E(G)| \geq 7|V(G)|$, then $\mathrm{cr}(G) \geq |E(G)|^3 / \big(29 \cdot |V(G)|^2\big)$.

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drawings, and by the Crossing Lemma for each of the drawings separately,

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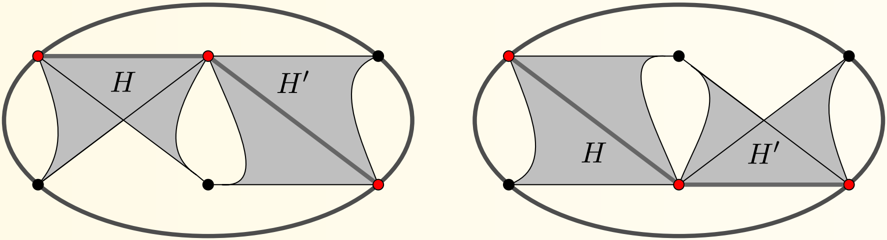
Proof (via a picture).



4 Complexity of Determining $\mathrm{ucr}(G)$

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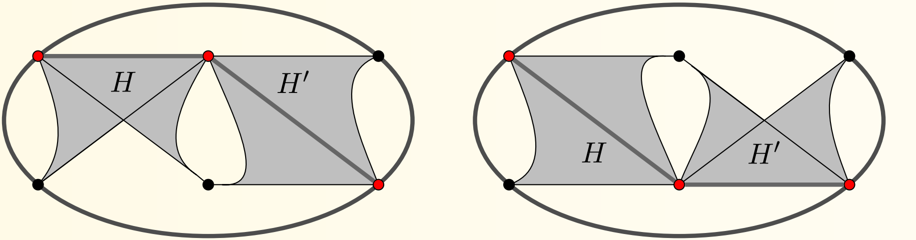

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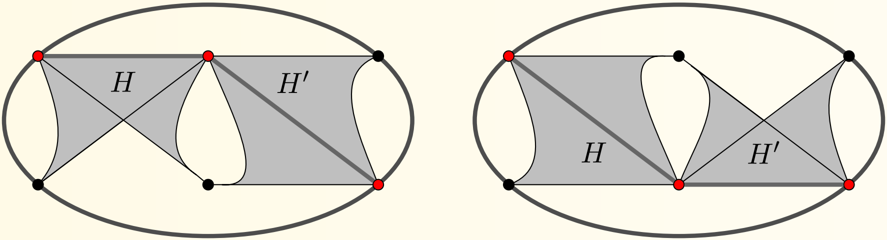

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Proof. Technical, very similar to classical Grohe's algorithm via MSO_2 logic...

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Thank you for your attention.