Vertex Insertion Approximates the Crossing Number for Apex Graphs*

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Abstract. We prove that the crossing number of an apex graph, i.e. a graph G from which only one vertex v has to be removed to make it planar, can be approximated up to a factor of $\Delta(G-v)\cdot d(v)/2$ by solving the *vertex inserting* problem, i.e. inserting a vertex plus incident edges into an optimally chosen planar embedding of a planar graph. Since the latter problem can be solved in polynomial time, this establishes the first polynomial fixed-factor approximation algorithm for the crossing number problem of apex graphs with bounded degree.
Furthermore, we extend this result by showing that the optimal solution for inserting multiple edges or vertices into a planar graph also approximates the crossing number of the resulting graph.

Keywords: crossing number, apex graph, vertex insertion, approximation.

1 Introduction

The *crossing number* $\mathrm{cr}(G)$ of a graph $G = (V, E)$ is the minimum number of pairwise edge crossings in a drawing of G in the plane. The crossing number problem has been vividly investigated for over 60 years, and yet only little is known about it. See [10] for an extensive bibliography. Even for seemingly simple graph classes, calculating—or at least bounding—the crossing number tends to be difficult. For example, we still only have conjectures for the crossing numbers of the complete and complete bipartite graphs. Determining the crossing number of a given graph is known to be NP-hard [5]. Even though there exist linear programming based exact algorithms which are promising for "real-world" graphs arising

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in practical graph drawing applications [3], computing exact crossing numbers is in general extremely difficult.

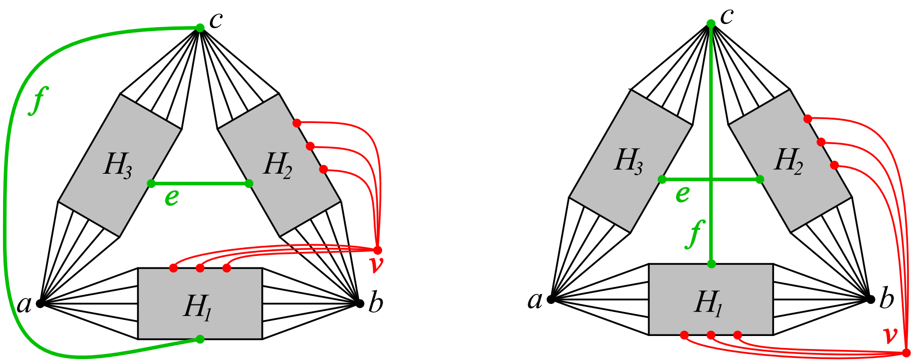
The best known polynomial algorithm for the crossing number of general graphs with bounded degree [4] approximates, within a factor of $\log^3 |V(G)|$, the quantity $|V(G)| + \mathrm{cr}(G)$, not directly $\mathrm{cr}(G)$. Perhaps the only currently known polynomial constant factor approximations of $\mathrm{cr}(G)$ are for projective [6], toroidal [9], and almost-planar graphs (see below), again assuming bounded degrees. On the other hand, the most common heuristic in practice is the *planarization method*: One starts with a planar subgraph G' and re-insert the temporarily removed edges one after another.

Let G' be a planar graph and $e \notin G'$ an edge not yet in G', connecting two nonadjacent vertices. *Inserting the edge e into the graph G'* means to find an embedding of G' and an insertion path for e, such that the resulting drawing induces a planar drawing of G' and has the smallest number of crossings. We denote this number by $\mathrm{ins}(G', e)$. While the complexity class of computing $\mathrm{cr}(G' + e)$ is unknown (and the weighted variant is NP-hard [1]), the computationally easier $\mathrm{ins}(G', e)$ clearly is an upper bound for $\mathrm{cr}(G' + e)$. We can summarize two main results regarding the edge insertion problem as follows:

Ins/1 Computing $\mathrm{ins}(G', e)$ can be done in linear time [7].
Ins/2 Optimally inserting a set of edges into a planar graph G' is NP-hard [11].

We say a non-planar graph G is *almost-planar* (also called *near planar* [1]) if it contains an edge e such that $G - e$ is planar. Given that the complexity of computing $\mathrm{cr}(G)$ is still unknown for almost-planar G, it was shown in [8] that $\mathrm{ins}(G - e, e)$ approximates $\mathrm{cr}(G)$. Recently, the best possible estimate $\mathrm{ins}(G - e, e) \leq \mathrm{cr}(G) \cdot \lfloor \Delta(G - e)/2 \rfloor$ has been proven in [1], whereby $\Delta(G - e)$ is the maximum degree in $G - e$. Hence the edge insertion algorithm in fact constitutes an approximation algorithm for the crossing number of almost-planar graphs which gives a fixed factor approximation in case of bounded degree.

In terms of insertion problems, the question arises which graph structures can be inserted optimally in polynomial time (similar to Ins/1), and when the structures become too complex (as for Ins/2). A natural generalization of the previous results is to consider the problem of *inserting a vertex v* with a specified neighborhood into a planar graph G' with the least number of crossings. We denote the latter number by $\mathrm{ins}(G', v)$. Although this shows to be a much harder question than that of edge insertion, it was recently shown that it is solvable in polynomial time [2].



(a) Solution of vertex insertion

(b) Optimal drawing

Fig. 1. An example of a vertex-v insertion instance requiring many crossings, even though the crossing number of the graph is small.

In this paper we, in turn, show that $\mathrm{ins}(G - v, v)$ approximates the crossing number of an *apex graph* G, i.e., a graph G with a vertex v whose removal leaves a planar graph. Our main result (see Section 2) reads:

Theorem 1. *Let G be a graph and v its vertex such that $G - v$ is planar. The vertex insertion problem of v into a planar embedding of $G - v$ has a solution with at most*

$$d(v) \cdot \lfloor \Delta(G - v)/2 \rfloor \cdot \mathrm{cr}(G) \text{ crossings,}$$

where $\Delta(G - v)$ is the maximum degree in $G - v$ and $d(v)$ is the degree of v in G.

Furthermore, powerful straightforward generalizations of this statement to multiple edge and vertex insertion problems are possible, and we state them later on in Theorems 7 and 8.

In connection with the algorithm [2] we hence immediately get the following.

Corollary 2. *There is a polynomial time algorithm that approximates the crossing number of an apex graph G (with apex vertex v) within a factor of at most $d(v) \cdot \lfloor \Delta(G - v)/2 \rfloor$. This is a fixed factor approximation in case of bounded degrees.*

A quite natural question arises; how far can the optimal solution to vertex insertion be from the crossing number? Inspired by the almost-planar constructions in [7, 8], we can give the following easy construction illustrated in Figure 1:

Lemma 3. *There exist apex graphs for which optimal solutions to the vertex insertion problem require up to $\lfloor d(v)/2 \rfloor \cdot \lfloor \Delta(G - v)/2 \rfloor \cdot \mathrm{cr}(G)$ crossings (about a half of the value in Theorem 1), for all values of $d(v)$ and $\Delta(G - v)$.*

Proof. Consider a graph G with a chosen node v as depicted in Figure 1. The gray regions denote large dense planar 3-connected subgraphs, i.e., they only allow a unique planar embedding or its mirror. Moreover, since they are "dense"—e.g., we may use sufficiently large grids—they have to be drawn planar in the considered optimal solutions and no edge will cross through such a subgraph; both would require many more crossings compared to the depicted solutions.

Roughly speaking, our graph mainly consists of three such dense planar 3-connected subgraphs (H_1, H_2, H_3), joined symmetrically via three high-degree nodes (a, b, c). Thus, each H_i ($1 \leq i \leq 3$) together with its two incident high-degree nodes forms a 3-connected component H'_i, with the high-degree nodes being the cut vertices of the graph. Each such component can be "flipped" (see Section 3 for a precise definition) along its cut nodes. We then augment this graph with two additional edges e, f and the vertex v (with incident edges) as depicted, forming the graph G.

Clearly, the removal of v from G leaves a planar graph. For the vertex insertion problem, $G-v$ has to be embedded planarly. We can achieve this by flipping $H'_1, ..., H'_3$ such that e and f can be drawn without any crossings (Figure 1(a)). In particular, since $G - v$ is 3-connected, this is the only possibility. Note that the former neighbors of v are now in two disjoint faces (i.e., regions bounded by edges), $\lfloor d(v)/2 \rfloor$ in each. Hence, inserting v into this embedding requires us to route $\lfloor d(v)/2 \rfloor$ edges between these two faces (inserting v anywhere along that route). Since any other route would be more expensive, this routing has to be "close to" b. Hence we obtain $\lfloor d(v)/2 \rfloor \cdot \lfloor \Delta(G - v)/2 \rfloor$ crossings.

Yet, by flipping H'_1 (Figure 1(b)), we do not require any crossings on the edges incident to v, but only 1 crossing between e and f. Since G is non-planar, the latter clearly is a crossing minimal solution. This establishes the claim. ∎

2 Crossing Number Approximation

The conceptual idea for proving Theorem 1 is based on [8]. But, in contrast to the former, we now require a more careful consideration of non-biconnected graphs, and the task is further complicated by the fact that the position of the newly introduced vertex v is unknown and possibly different between the solution of the insertion and the crossing number problems.

Assume Γ is a plane embedding of the graph $G-v$ achieving optimality in the vertex-v insertion problem, Γ_c is a crossing-optimal drawing of the graph G, and let F be a suitable minimal set of edges such that $\Gamma_c - v - F$ is a plane embedding.

Then $|F| \leq \mathrm{cr}(G)$ and the embedding $\Gamma_c - v - F$ can be turned into $\Gamma - F$ by a sequence of 1- and 2-flips (again, see Section 3 for the precise definition), which consequently allows to re-embed the edges of F without crossings in $G - v$. In this situation the number of new crossings introduced on the edges of v can be bounded by an iteration of the following claim over all $f \in F$:

Lemma 4. *Let H be an apex graph with an apex vertex v such that $H - v$ is connected. Let an edge f connect two (nonadjacent) vertices of $H - v$. If $(H - v) + f$ is planar, then there is a drawing of $H + f$ with plane embedded $(H - v) + f$ having at most $\mathrm{ins}(H - v, v) + d(v) \cdot \lfloor \Delta(H - v)/2 \rfloor$ crossings.*

We will prove this lemma in the next section. By using it, we now establish our main Theorem 1. Although an application of the previous iteration scheme seems straightforward, it is not so due to the unavoidable requirement for connected $H - v$ in Lemma 4—notice that for an arbitrary minimal F as above, the graph $G - v - F$ might (easily) become disconnected. The solution is a careful choice of the edges in F to maintain as much connectivity as possible.

Proof of Theorem 1. Let Γ and Γ_c be defined as above. Notice, first of all, that in degenerate cases of $d(v) \leq 1$ or $\Delta(G - v) \leq 2$ there always is a solution with $\mathrm{ins}(G - v, v) = 0$. Otherwise, we proceed our proof by induction on $\mathrm{cr}(G)$.

If $\Gamma_c - v$ is a plane drawing, then we have a solution with $\mathrm{ins}(G - v, v) = \mathrm{cr}(G)$.

A *bridge* is an edge whose removal would disconnect the corresponding graph. So, assume $\Gamma_c - v$ contains a crossing involving an edge f of $G - v$ such that f is not a bridge in $G - v$. Setting $H := G - f$, we see that $\mathrm{cr}(H) \leq \mathrm{cr}(G) - 1$ from crossing-optimality of our Γ_c. By inductive assumption, with H in place of G, we obtain

$$\mathrm{ins}(H - v, v) \leq d(v) \cdot \lfloor \Delta(H - v)/2 \rfloor \cdot \mathrm{cr}(H)\,,$$

and by immediately subsequent application of Lemma 4,

$$\begin{aligned} \mathrm{ins}(G - v, v) &\leq d(v) \cdot \lfloor \Delta(H - v)/2 \rfloor \cdot (\mathrm{cr}(G) - 1) + d(v) \cdot \lfloor \Delta(H - v)/2 \rfloor \leq \\ &\leq d(v) \cdot \lfloor \Delta(G - v)/2 \rfloor \cdot \mathrm{cr}(G)\,. \end{aligned}$$

It thus remains to consider that all the edges involved in crossings in $\Gamma_c - v$ are bridges of $G - v$ (hence forming a "tree-like" structure with the remaining components as "big vertices"). Let F_0 be any minimal set of edges in $G - v$ covering all the crossings in $\Gamma_c - v$, i.e., for each crossing, at least one involved edge is contained in F_0. In particular, $|F_0| \leq \mathrm{cr}(G)$.

If $f \in F_0$, then both ends of f belong to the same face of $\Gamma_c - v - F_0$; otherwise these ends would be separated by a cycle C in $\Gamma_c - v - F_0$ and the edge of C crossing f would not be a bridge. Hence in this case we can iteratively re-insert the edges of F_0 back to $\Gamma_c - F_0$, each time crossing at most $d(v)$ edges incident with v and no other edge of Γ_c. So (without using induction here), we obtain

$$\mathrm{ins}(G - v, v) \leq (\mathrm{cr}(G) - |F_0|) + |F_0| \cdot d(v) \leq d(v) \cdot \mathrm{cr}(G) .$$

∎

3 Proof of Lemma 4

It remains to prove Lemma 4. Therefore we will simplify our notation and consider only the planar graphs $G := H - v$, and $G + f$ where f connects two nonadjacent vertices x, y of G. Let $\Delta = \Delta(G)$. Let $w_1, \dots, w_d$ be the d former neighbors of v in G (in H, in fact), thereafter called the *"red terminals"* of G. We no longer treat v and its former edges from H as a graph vertex and edges, but as the *"red point"* v and the *"red lines"* drawn from v to the terminals $w_1, \dots, w_d$ inside a plane embedding of G or of $G + f$. Hence a *"crossing"* is for us now a crossing between a red line and an edge of some embedding of G. No other kinds of crossings will occur, as we will see later. The following is a reformulation of Lemma 4 in this special setting:

Lemma 5 (Alternative Formulation of Lemma 4). *Let Γ be a plane embedding of a connected graph G, and $x, y \in V(G)$ be such that $G + xy$ is planar. Then there exists a plane embedding Γ' of G such that:*

a) The vertices x, y belong to the same face of Γ', i.e. the edge $f = xy$ can be planarly inserted into Γ'.

b) Assume we can draw a red point joined by red lines to all the terminals $w_1, \dots, w_d \in V(G)$ in Γ with ℓ crossings. Then the same can be drawn into $\Gamma' + f$ with at most $\ell + d \cdot \lfloor \Delta/2 \rfloor$ crossings.

We need some more technical terms and conditions before proceeding with the proof.

Let $G \restriction A$ and $\Gamma \restriction A$ denote the subgraph of G and the subembedding of Γ, respectively, induced by the edges $A \subseteq E(G)$. We use the *bar*-notation $\bar{A} := E(G) \setminus A$ to specify the complement of some edge set $A \subseteq E(G)$ w.r.t. $E(G)$. A *k-separation* in a graph G is a bipartition $(A, \bar{A})$ of the edges $E(G)$ such that exactly k *boundary* vertices of G are incident both with A and $\bar{A}$.

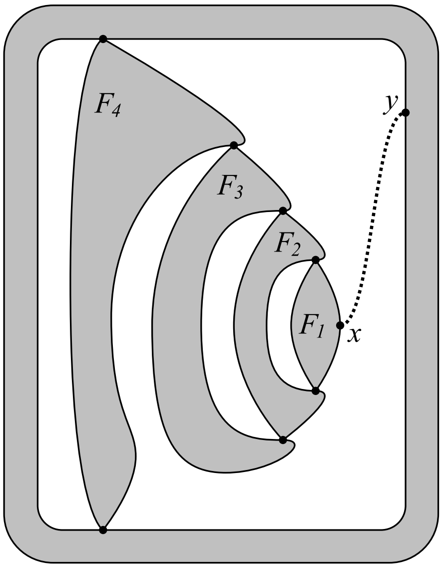


Fig. 2. Lemma 6: A nested sequence of 1- and 2-flips allows to obtain a planar embedding where x and y are on the same face. The gray regions denote subgraphs; note that each subgraph $G \restriction F_i$ contains the subgraph $G \restriction F_{i-1}$.

This is a technical analogy of a k-vertex cut. Having a 2-separation $(A, \bar{A})$ in G, and an embedding Γ of G, a 2-*flip* of A in Γ (also called a Whitney flip) is the operation which "cuts out" the subembedding $\Gamma \restriction A$ and embeds back its mirror image, exactly matching the former two cutvertices of $(A, \bar{A})$. A 2-flip is a well-known graph operation, and a folklore theorem by Whitney states that any two plane embeddings of a 2-connected planar graph are *equivalent* up to a sequence of 2-flips. Two plane embeddings are equivalent if they have the same collections of facial cycles (or of facial walks in the case of a connected graph).

Unlike the former, no established notion of a "1-flip" seems to exist. The following notion is suited to our needs: Having a 1-separation $(A, \bar{A})$ in G with the cutvertex z, and an embedding Γ of G, a 1-*flip* of A in Γ is the operation which "cuts out" the subembedding $\Gamma \restriction A = \Gamma_A$, then makes any face of Γ_A incident with z the new outer face of Γ_A, and finally embeds Γ_A or its mirror image back to any face of $\Gamma \restriction \bar{A}$ incident with z again. Note that a 1-flip operation on A is not uniquely determined, and that our definition is actually symmetric in the parts $A, \bar{A}$. We shall use the following technical statement, cf. Figure 2:

Lemma 6. *Let Γ, a plane embedding of G, and $x, y \in V(G)$ be defined as in Lemma 5. Then there exists an sequence $F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_t \subsetneq E(G)$ of edge sets such that:*

a) Vertex x is only incident with edges of F_1, vertex y is not incident with any edge of F_t, and $\left(F_i, \bar{F_i}\right)$ is a 1- or 2-separation in G for $i = 1, \ldots, t$.

b) The subembedding $\Gamma \restriction F_i$ is contained in a single face of $\Gamma \restriction \bar{F}_i$, and symmetrically $\Gamma \restriction \bar{F}_i$ is contained in a single face of $\Gamma \restriction F_i$, for $i = 1, \ldots, t$.

c) Successively applying 1- or 2-flips of F_i, $i = 1, \ldots, t$, onto Γ leads to an embedding Γ_0 of G such that the vertices x, y belong to the same face of Γ_0.

Proof. Let Γ_f be any plane embedding of the graph $G+f$ where $f = xy$. We proceed the proof by induction on the number of blocks (2-connected components) of G.

- If G itself is 2-connected, then Γ can be transformed into $\Gamma_f - f$ using a sequence $\mathcal{S}$ of 2-flips by Whitney's theorem. These flips clearly commute. If a flip of $F \subset E(G)$ is in the sequence $\mathcal{S}$ such that $x, y \in G \restriction F$ or $x, y \in G \restriction \bar{F}$, then it can be undone later without affecting the embeddability (c) of the new edge f. Hence we can eliminate all such flips in advance. Possibly taking set complements, we can thus assume that flipping of F occurs in our sequence $\mathcal{S}$ only if x is incident exclusively with edges of F and y is not incident with any edge of F. This establishes claim (a).
If the sequence $\mathcal{S}$ considers two overlapping 2-flips of F and of F', i.e. both $F \setminus F'$ and $F' \setminus F$ are nonempty, then also $F \cap F'$ and $F \cup F'$ are 2-separations in G and we can instead flip those two sets. Hence we may assume that our 2-flipping sequence transforming Γ to suitable Γ_f with added edge f is of the form $F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_t \subsetneq E(G)$. Finally, to deal with the technical condition (b), we notice that if the two boundary vertices z_1, z_2 of any 2-separation in G give more than two components in $G - z_1 - z_2$, then only two of these components containing x and y are interesting for inserting f. The remaining ones can be flipped with either side such that (b) is satisfied.
- It remains to consider a non-2-connected graph G. If a leaf block of G is incident with neither x, y, then this block can be safely ignored and we proceed by induction without this block. Hence consider a leaf block K of G with a cutvertex z, such that x is disjoint from K and y belongs to $K - z$; let $C = E(K)$. By inductive assumption, there is a flipping sequence $F_1^1 \subsetneq \ldots \subsetneq F_{t_1}^1 \subsetneq \bar{C}$ transforming $\Gamma \restriction \bar{C}$ into Γ_1 such that the vertices x and z are on the same face of Γ_1. Similarly a flipping sequence $F_1^2 \subsetneq \ldots \subsetneq F_{t_2}^2 \subsetneq C$ transforms $\Gamma \restriction C$ into Γ_2 such that the vertices z and y are on the same face of Γ_2. Clearly, an appropriate 1-flip of $\bar{C}$ now transforms $\Gamma_1 \cup \Gamma_2$ into an embedding of G in which all three vertices x, z, y are on the same face. Hence we conclude with a composed flipping sequence $F_1^1 \subsetneq \ldots \subsetneq F_{t_1}^1 \subsetneq$

$\bar{C} \subsetneq F_1^2 \cup \bar{C} \subsetneq \ldots \subsetneq F_{t_2}^2 \cup \bar{C} \subsetneq E(G)$ which again satisfies the conditions of the lemma. ∎

Proof of Lemma 5. Without loss of generality, we assume that the face of Γ hosting the red point of part (b) is the outer face. We consider the edge-set sequence $F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_t \subsetneq E(G)$ given by Lemma 6.

Let $j \in \{1, \ldots, t\}$ be the smallest index such that at least one edge of F_j is incident with the outer face of Γ. If all the edges of the outer face belong to $\bar{F}_t$, then let $j = t + 1$. We set $p = j - 1$ and $q = t + 1 - j$, and define two edge-set sequences $A_i = F_i$ for $i = 1, \ldots, p$, and $B_i = \bar{F}_{t+1-i}$ for $i = 1, \ldots, q$. Let $A_0 = B_0 = \emptyset$, and $D = E(G) \setminus (A_p \cup B_q)$. Now, successively applying appropriate 1- or 2-flips on all these A_i and B_i (while keeping $\Gamma \restriction D$ fixed) lead to an embedding Γ' of G that is equivalent to Γ_0 of Lemma 6, and hence x, y belong to the same face of Γ' and $\Gamma' + f$ is plane. This establishes claim (a).

Later on, we shall use another fact implied by this situation. Let Ω be the face of $\Gamma \restriction D$ containing $\Gamma \restriction A_p$. We claim that Ω must also contain $\Gamma \restriction B_q$: We have $\Gamma \restriction D = \Gamma' \restriction D$, and the vertices x in $\Gamma \restriction A_p$ and y in $\Gamma \restriction B_q$ do not belong to $\Gamma \restriction D$. Planarity of $\Gamma' + f$ forces x and y to occur simultaneously in Ω. Furthermore, $\Gamma \restriction A_p$ is contained in the outer face of $\Gamma \restriction B_q$ by our choice of j above, and symmetrically $\Gamma \restriction B_q$ is in the outer face of $\Gamma \restriction A_p$ using Lemma 6(b).

To prove the more difficult part (b) of Lemma 5, we start from the optimal "red drawing" (joining a red point with all the red terminals $w_1, \ldots, w_d$) in Γ. We then suitably modify the red lines incident with terminals involved in the flipping sequences A_i or B_i: We bring them "close from outside" to the vertex x or y, respectively, and finally re-join all these red lines in the face of Γ' hosting the inserted edge $f = xy$. Our general goal is to add at most $\lfloor \Delta/2 \rfloor$ new crossings on each of the d red lines; cf. Figure 3.

Formally, a terminal w_j is *involved* in the flip of A_i if all edges incident with w_j in G belong to A_i; the analogous holds for B_i.

Consider the original Γ together with an optimal drawing of a red point v joined by red lines to all the terminals $w_1, \ldots, w_d \in V(G)$, altogether requiring ℓ crossings. We call this one the *old* red drawing, to distinguish it from a *new* red drawing we are going to construct in Γ'. By our assumptions above, v is in the outer face Λ of Γ, but all the terminals involved in our two flipping sequences are contained in Ω, the previously defined face of $\Gamma \restriction D$.

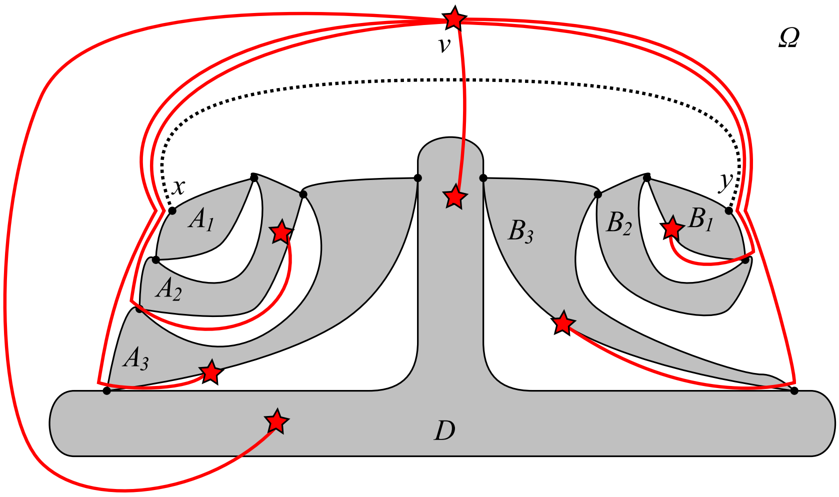


Fig. 3. Lemma 5: After realizing the flipping sequences A_i and B_i, we can planarly insert $f = xy$. The stars within the graph denote the red terminals. Re-routing the red (bold) lines through the neighborhood of to the cut nodes and bringing the lines close to x or y allows us to bound the number of additional crossings on these lines.

For reference we denote by Γ_i the embedding obtained from Γ after applying the subsequence of appropriate flips on $A_1, \ldots, A_i$, for $i = 1, \ldots, p$. Then x will always be on the outer face of $\Gamma_i \restriction A_i$ since otherwise x could not be later planarly joined with y belonging to $\Gamma' \restriction B_q$. Furthermore, if a red terminal w_j gets involved in the flipping of A_i (but not in the flipping of A_{i-1}), then a segment of the old red line in Γ joining w_j to red v will also join w_j to the outer face of $\Gamma_{i-1} \restriction A_i$ without additional crossings. Let α_j denote this red line segment from w_j. Notice, however, that the outer face of $\Gamma_{i-1} \restriction A_i$ may be different from the outer face of the next $\Gamma_i \restriction A_i$ in case of 1-flip.

The previous considerations suggest the following procedure leading to a new "red drawing" within $\Gamma' + f$, as required by the part (b). We note in advance that possible red-red crossings (i.e. between two red lines in the coming new red drawing) will not matter since all the red lines come from a central point v. Any such crossing can be later eliminated (at no additional cost) by mutually exchanging the red sections between v and that crossing.

- For $i = 1, \ldots, p$, the appropriate 1- or 2-flip of A_i (cf. Lemma 6(c)) is applied to Γ in such a way that the red line segments incident with the involved red terminals in A_i get flipped together with $\Gamma_{i-1} \restriction A_i$. Consider a red terminal w_j (independently of the others) that has not been involved in the flip of A_{i-1}, but is involved in the flip of A_i. Its red segment α_j is flipped with A_i.

We draw the new red line from w_j using this α_j which we extend such that it reaches a close neighborhood of the vertex x on the outer face of $\Gamma_i \restriction A_i$:

 - If A_i is a 2-separation with the boundary $\{z_1, z_2\}$, then the outer face of $\Gamma_{i-1} \restriction A_i$ (in which α_j ends so far) is the same as the outer face of $\Gamma_i \restriction A_i$. There are, though, two faces of Γ_i incident with A_i and contained in the outer face of $\Gamma_i \restriction A_i$ (note that these two faces meet in z_1 and z_2). Bringing α_j to a close neighborhood of x may require "passing closely by" the vertex z_1 or z_2. Hence in this (one-time) situation we require either 1 future crossing of α_j with the edge f, or at most $\max\{\lfloor d(z_1)/2\rfloor, \lfloor d(z_2)/2\rfloor\} \leq \lfloor \Delta/2\rfloor$ crossings with edges incident to z_1 or z_2. By correctly choosing between z_1 and z_2, we can circumvent any future crossings between α_j and f in the latter case.
 - If A_i is a 1-separation with the boundary $\{z_3\}$, then α_j has reached a face of $\Gamma_i \restriction A_i$ incident with z_3, but this face may not be the outer face (with x). We can again bring α_j to a close neighborhood of x by closely passing around z_3 at cost of at most $\lfloor d(z_3)/2\rfloor \leq \lfloor \Delta/2\rfloor$ crossings with edges incident to z_3, and we can again easily avoid a future crossing of f by symmetry around z_3.

- We can apply the same construction symmetrically and independently to the other flipping sequence of $B_1, \ldots, B_q$ and its involved red terminals.
- At the end of previous constructions, there is the embedding Γ' of G, and a face $\Omega_0 \subset \Omega$ of Γ' hosting both vertices x and y. Furthermore, for each red terminal w_j that is involved in the flipping sequences, we have a new red α_j-segment starting in w_j and ending in Ω_0 in a close neighborhood of x or y, respectively. Let Ω_1 denote the common intersection of Ω, of the outer face of $\Gamma' \restriction A_p$, and of the outer face of $\Gamma' \restriction B_q$. Then $\Omega_0 \subseteq \Omega_1$, but the full Ω_1 consists of up to three faces of Γ'. Generally, to extend a red line from Ω_0 to any of the other two faces in Ω_1, or vice versa, one can pass closely along some z, a boundary vertex of the separation of A_p or B_q, at cost of $\lfloor d(z)/2\rfloor \leq \lfloor \Delta/2\rfloor$ additional crossings.

In contrast to the *involved* terminals we say a terminal is *untouched* if is was not involved in flips of any A_i or B_i. We have to consider the following three cases:

 - Ω is the outer face of Γ' containing red v (that is the case depicted in Figure 3). In that case we choose a new red point v' in a close neighborhood of f, and directly connect all the new red α_j-segments of the involved terminals to v'. Then, since the old red lines of all untouched terminals must enter Ω due to the old v, we can prolong also those lines towards v' in Ω_0 as discussed above. Overall, every new red line requires at most $\lfloor \Delta/2 \rfloor$ additional crossings, and the new red drawing thus has at most $\ell + d \cdot \lfloor \Delta/2 \rfloor$ crossings with edges of Γ'.

- Ω does not contain the red v, but the number of untouched red terminals is at most the number of involved red terminals. Let β denote a section of some old red line joining v to Ω with the least number of crossings with $\Gamma \restriction D$. We first pull all the red lines of the untouched red terminals along β towards Ω. The number of additional crossings with $\Gamma \restriction D$ on those new red lines is, by our assumption, not bigger than the number of crossings on the old red lines joining v to the involved terminals. Those old crossings will now become eliminated. We finish as in the previous case.
- Finally, Ω does not contain the red v, and the number of involved red terminals is $c < d/2$ (i.e. there are fewer involved than untouched terminals). We choose the new red v' identical to the old v (thus not requiring any new crossings for the red lines of untouched terminals). Let β be defined as above. We extend all the c new red α_j-segments of the involved terminals towards the end of β in Ω, which can cost up to additional $\lfloor \Delta/2 \rfloor$ crossings on each α_j-segment as discussed above. Then we prolong all those lines along β to $v' = v$ which adds here no more crossings than in the old red drawing—every old red line to an involved terminal has crossed $\Gamma \restriction D$ at least as many times as β does. Hence in this case, the new red drawing has at most $\ell + c \cdot \lfloor \Delta/2 \rfloor + c \cdot \lfloor \Delta/2 \rfloor \leq \ell + d \cdot \lfloor \Delta/2 \rfloor$ crossings with edges of Γ'.

Lemma 5, and therefore also Lemma 4 and Theorem 1, are now proven. ∎

4 Conclusions

We have shown that the vertex insertion problem finds an approximate solution for the crossing number problem of any apex graph G (with an apex vertex v), which is at most a factor of $d(v) \cdot \lfloor \Delta(G - v)/2 \rfloor$ away from $\mathrm{cr}(G)$. Yet, we can only give an example requiring half of this factor in Proposition 3 (Figure 1) . It remains an open question whether our bound can be improved by this difference.

Our proving strategy builds upon the one devised in [8] for the edge insertion problem. The approximation factor given therein has been later halved (obtaining a tight factor) by Cabello and Mohar [1], using an alternative proving strategy. We feel that also in the case of vertex insertion, the actual bound of our approximation algorithm should in fact be the one required by our construction in Proposition 3. The strategy of [1] builds upon the concept of *facial distance* between the two nodes that are to be connected via the inserted edge. It is unclear how this concept could be generalized in the context of vertex insertion.

Although the problem of optimally inserting multiple edges (simultaneously) into a planar graph is NP-hard, it is remarkable that we can generalize our proof to show that $\mathrm{ins}(G - E', E')$—the number of crossings necessary to insert the edges E' into a planar embedding of $G - E'$, approximates the crossing number $\mathrm{cr}(G)$, too.

Theorem 7. *Let G be a graph and E' a subset of its edges such that $G - E'$ is planar. Then* $\mathrm{ins}(G - E', E') \leq |E'| \cdot \Delta(G - E') \cdot \mathrm{cr}(G) + \binom{|E'|}{2}$.

We do not give a separate proof of this statement since it is analogous to the proof of Theorem 8 below.

We can go even further and consider a more general *multiple-vertex insertion problem*, asking for the number $\mathrm{ins}(G - V', V')$ of crossings necessary to re-insert the (independent) vertices V' into a planar embedding of $G - V'$.

Theorem 8. *Let G be a graph and $V' = \{v_1, \ldots, v_m\}$ an independent subset of its vertices such that $G - V'$ is planar. Then*

$$\mathrm{ins}(G - V', V') \leq \left(\sum_{i=1}^{m} d(v_i) \right) \cdot \lfloor \Delta(G - V')/2 \rfloor \cdot \mathrm{cr}(G) + \sum_{1 \leq i < j \leq m} d(v_i) d(v_j) \,.$$

Proof. Let $G' = G - V'$. For each $i = 1, 2, \ldots, m$ independently, we apply the arguments of Theorem 1 to the vertex-v_i insertion problem for planar G', and hence obtain a particular solution with at most $d(v_i) \cdot \lfloor \Delta(G')/2 \rfloor \cdot \mathrm{cr}(G' + v_i)$ crossings. The crucial fact is that our proofs of Theorem 1 and mainly of Lemma 5 never query positions of the "red" terminals of v_i for the purpose of drawing the final plane embedding of G'. So, all the independent solutions to the vertex-v_i insertions share equivalent plane subembeddings of G' at the end. (In the language of Section 3, we can thus draw one plane embedding of G' simultaneously with a bunch of red lines from v_1, a bunch of orange lines from v_2, a bunch of pink lines from v_3, and so on. . .)

Therefore, it is possible (rigorously) to combine those particular solutions into the desired conclusion

$$\mathrm{ins}(G - V', V') \leq \sum\nolimits_{i=1}^{m} d(v_i) \cdot \lfloor \Delta(G')/2 \rfloor \cdot \mathrm{cr}(G' + v_i) + \Psi \leq$$

$$\leq \left(\sum_{i=1}^{m} d(v_i)\right) \cdot \lfloor \Delta(G')/2 \rfloor \cdot \mathrm{cr}(G) + \Psi\,,$$

where an additive factor of $\Psi = \sum_{1\leq i<j\leq m} d(v_i)d(v_j)$ expresses the fact that we cannot generally prevent crossings (at most one per pair) between edges incident with v_i and with v_j for $i \neq j$. ∎

So, finally, since the multiple-edge and multiple-vertex insertion problems are NP-hard, how can Theorems 7 and 8 help with solving the crossing number problem? We hope that, at least in some special settings, the multiple-edge or multiple-vertex insertion problems could be approximated in polynomial time. This will then automatically give approximation algorithms for the corresponding crossing number problems.

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