Twin-width of Planar Graphs
a Short Proof

Petr Hliněný

Faculty of Informatics, Masaryk University
Brno, Czech Republic

1 Trigraphs and Twin-Width

Definition. A *trigraph* is just a simple graph with some edges marked red.

1 Trigraphs and Twin-Width

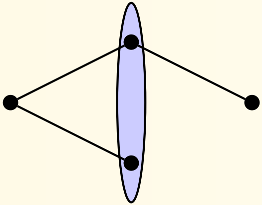
Definition. A *trigraph* is just a simple graph with some edges marked red.

– Red edges signal discrepancies / errors in the adjacency relation during (and after) vertex-pair contractions.

1 Trigraphs and Twin-Width

Definition. A *trigraph* is just a simple graph with some edges marked red.

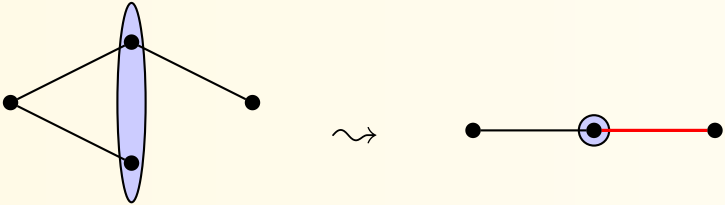
– Red edges signal discrepancies / errors in the adjacency relation during (and after) vertex-pair contractions.



1 Trigraphs and Twin-Width

Definition. A *trigraph* is just a simple graph with some edges marked red.

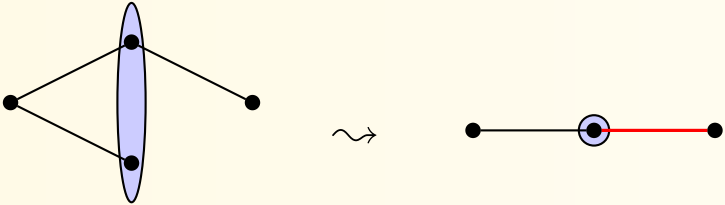
– Red edges signal discrepancies / errors in the adjacency relation during (and after) vertex-pair contractions.



1 Trigraphs and Twin-Width

Definition. A *trigraph* is just a simple graph with some edges marked red.

– Red edges signal discrepancies / errors in the adjacency relation during (and after) vertex-pair contractions.

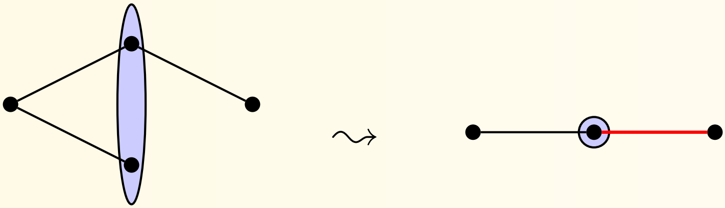


Definition. The **twin-width** of a simple graph G is the least integer d

1 Trigraphs and Twin-Width

Definition. A *trigraph* is just a simple graph with some edges marked red.

– Red edges signal discrepancies / errors in the adjacency relation during (and after) vertex-pair contractions.

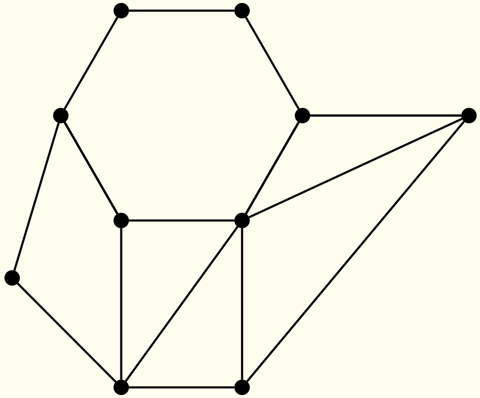


Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.

[Bonnet, Kim, Thomassé and Watrigant, FOCS 2020]

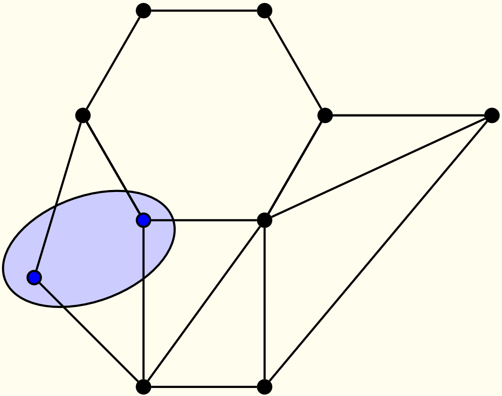
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



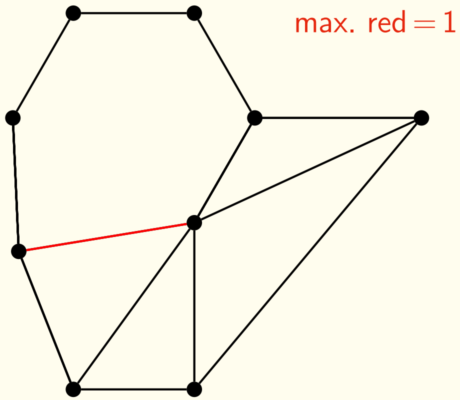
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



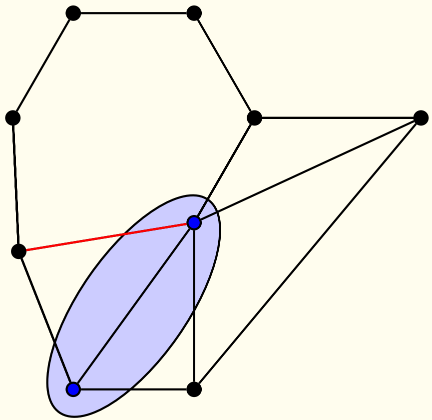
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



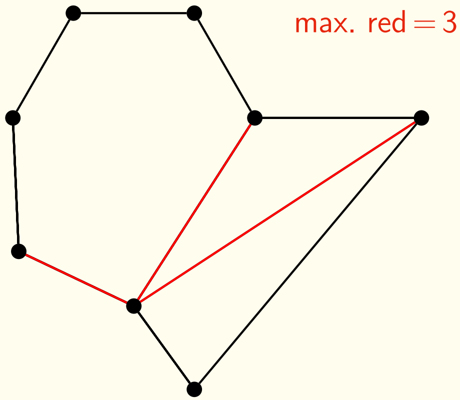
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



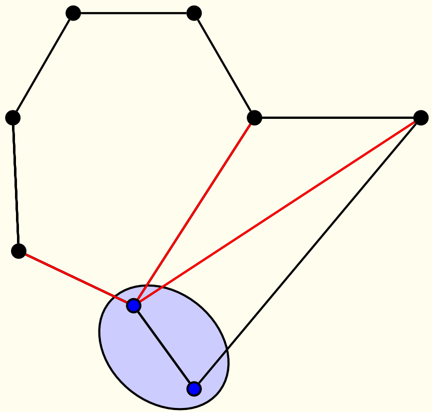
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



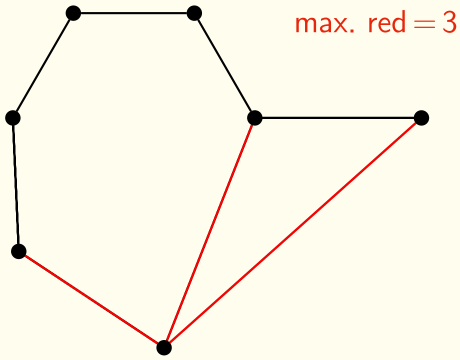
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



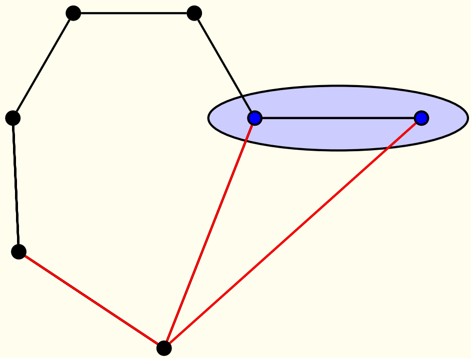
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



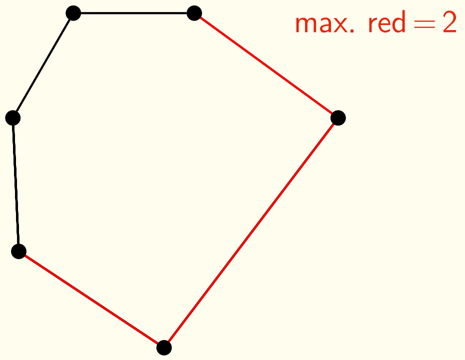
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



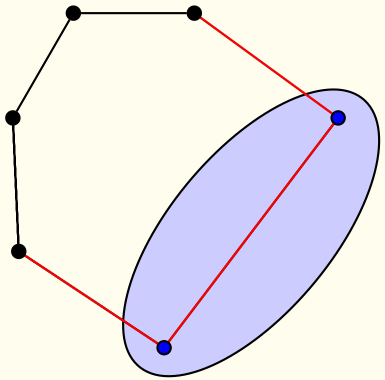
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



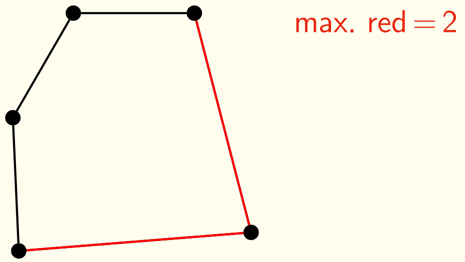
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



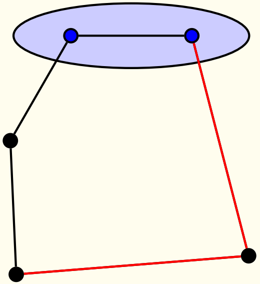
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



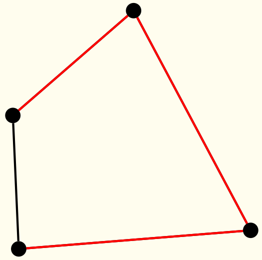
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



Twin-Width Example

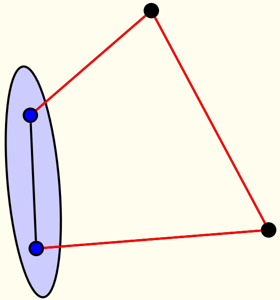
Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



max. red $= 2$

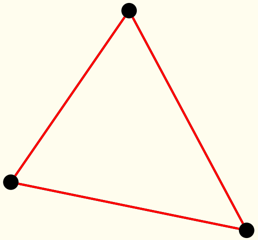
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



Twin-Width Example

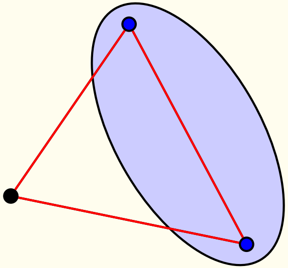
Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



max. red $= 2$

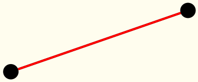
Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



Twin-Width Example

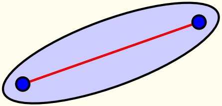
Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



max. red $= 1$

Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.



Twin-Width Example

Definition. The **twin-width** of a simple graph G is the least integer d such that there exists a contraction sequence of G in which every *trigraph* has maximum red degree $\leq d$.

max. red $= 0$

twin-width ≤ 3

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].
- ArXiv Feb 2022: $\leq \mathbf{583}$ by Bonnet, Kwon and Wood.

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].
- ArXiv Feb 2022: $\leq \mathbf{583}$ by Bonnet, Kwon and Wood.
- ArXiv Apr 2022: $\leq \mathbf{37}$ by Bekos, Da Lozzo, PH and Kaufman.

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].
- ArXiv Feb 2022: $\leq \mathbf{583}$ by Bonnet, Kwon and Wood.
- ArXiv Apr 2022: $\leq \mathbf{37}$ by Bekos, Da Lozzo, PH and Kaufman.
- ArXiv Jun 2022: $\leq \mathbf{9}$ by PH.

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].
- ArXiv Feb 2022: $\leq \mathbf{583}$ by Bonnet, Kwon and Wood.
- ArXiv Apr 2022: $\leq \mathbf{37}$ by Bekos, Da Lozzo, PH and Kaufman.
- ArXiv Jun 2022: $\leq \mathbf{9}$ by PH.
- ArXiv Sep 2022: $\geq \mathbf{7}$ by Král' and Lamaison.

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].
- ArXiv Feb 2022: $\leq \mathbf{583}$ by Bonnet, Kwon and Wood.
- ArXiv Apr 2022: $\leq \mathbf{37}$ by Bekos, Da Lozzo, PH and Kaufman.
- ArXiv Jun 2022: $\leq \mathbf{9}$ by PH.
- ArXiv Sep 2022: $\geq \mathbf{7}$ by Král' and Lamaison.
- ArXiv Oct 2022: $\leq \mathbf{8}$ by PH and Jedelský [ICALP'23].

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].
- ArXiv Feb 2022: $\leq \mathbf{583}$ by Bonnet, Kwon and Wood.
- ArXiv Apr 2022: $\leq \mathbf{37}$ by Bekos, Da Lozzo, PH and Kaufman.
- ArXiv Jun 2022: $\leq \mathbf{9}$ by PH.
- ArXiv Sep 2022: $\geq \mathbf{7}$ by Král' and Lamaison.
- ArXiv Oct 2022: $\leq \mathbf{8}$ by PH and Jedelský [ICALP'23].

So, what NEW?

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].
- ArXiv Feb 2022: $\leq \mathbf{583}$ by Bonnet, Kwon and Wood.
- ArXiv Apr 2022: $\leq \mathbf{37}$ by Bekos, Da Lozzo, PH and Kaufman.
- ArXiv Jun 2022: $\leq \mathbf{9}$ by PH.
- ArXiv Sep 2022: $\geq \mathbf{7}$ by Král' and Lamaison.
- ArXiv Oct 2022: $\leq \mathbf{8}$ by PH and Jedelský [ICALP'23].

So, what NEW? With a short and elementary proof;

- **The twin-width of any simple planar graph is at most 11.**

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].
- ArXiv Feb 2022: $\leq \mathbf{583}$ by Bonnet, Kwon and Wood.
- ArXiv Apr 2022: $\leq \mathbf{37}$ by Bekos, Da Lozzo, PH and Kaufman.
- ArXiv Jun 2022: $\leq \mathbf{9}$ by PH.
- ArXiv Sep 2022: $\geq \mathbf{7}$ by Král' and Lamaison.
- ArXiv Oct 2022: $\leq \mathbf{8}$ by PH and Jedelský [ICALP'23].

So, what NEW? With a short and elementary proof;

- **The twin-width of any simple planar graph is at most 11.**

And the right exact answer? 7 or 8?

Bounding Twin-Width of Planar Graphs

- Astronomical upper bounds already since the first pap. [FOCS 2020].
- ArXiv Jan 2022: $\leq \mathbf{183}$ by Jacob and Ma. Pilipczuk [WG'22].
- ArXiv Feb 2022: $\leq \mathbf{583}$ by Bonnet, Kwon and Wood.
- ArXiv Apr 2022: $\leq \mathbf{37}$ by Bekos, Da Lozzo, PH and Kaufman.
- ArXiv Jun 2022: $\leq \mathbf{9}$ by PH.
- ArXiv Sep 2022: $\geq \mathbf{7}$ by Král' and Lamaison.
- ArXiv Oct 2022: $\leq \mathbf{8}$ by PH and Jedelský [ICALP'23].

So, what NEW? With a short and elementary proof;

- **The twin-width of any simple planar graph is at most 11.**

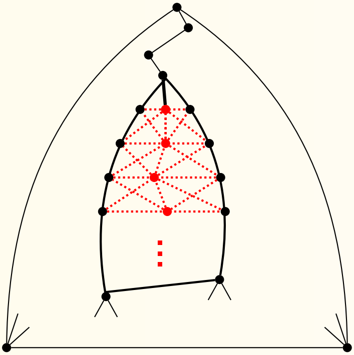
And the right exact answer? 7 or 8?
7 hopefully soon. . .

2 Basic Setup for the Proof(s)

- A trigraph G with a *skeleton* $S \subseteq G$, where S is a plane 2-connected black graph not crossed by any edges of $E(G) \setminus E(S)$.

2 Basic Setup for the Proof(s)

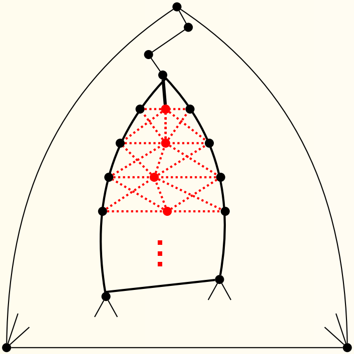
- A trigraph G with a *skeleton* $S \subseteq G$, where S is a plane 2-connected black graph not crossed by any edges of $E(G) \setminus E(S)$.



2 Basic Setup for the Proof(s)

- A trigraph G with a *skeleton* $S \subseteq G$, where S is a plane 2-connected black graph not crossed by any edges of $E(G) \setminus E(S)$.

- A BFS tree $T \subseteq G$ such that every S-face containing vertices of $V(G) \setminus V(S)$ is bounded by two T-paths $+$ edge.

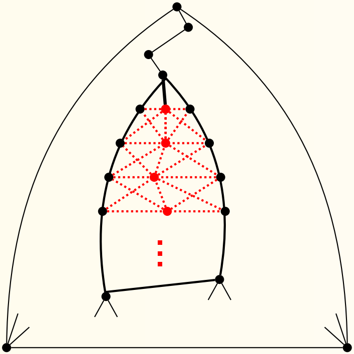


2 Basic Setup for the Proof(s)

- A trigraph G with a *skeleton* $S \subseteq G$, where S is a plane 2-connected black graph not crossed by any edges of $E(G) \setminus E(S)$.

- A BFS tree $T \subseteq G$ such that every S-face containing vertices of $V(G) \setminus V(S)$ is bounded by two T-paths $+$ edge.

- Contractions respect the BFS layers, and stick with ≤ 3 vert. per S-face and layer.



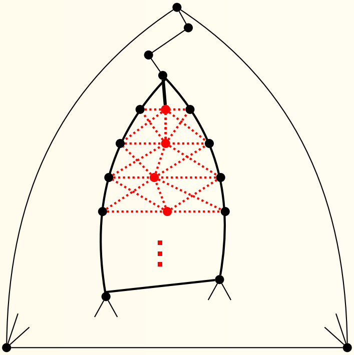
2 Basic Setup for the Proof(s)

- A trigraph G with a *skeleton* $S \subseteq G$, where S is a plane 2-connected black graph not crossed by any edges of $E(G) \setminus E(S)$.

- A BFS tree $T \subseteq G$ such that every S-face containing vertices of $V(G) \setminus V(S)$ is bounded by two T-paths $+$ edge.

- Contractions respect the BFS layers, and stick with ≤ 3 vert. per S-face and layer.

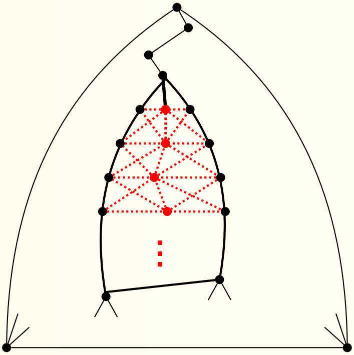
- $V(S)$ untouched by contractions, and so edges induced by $V(S)$ are black, other (except at the sink) considered red.



2 Basic Setup for the Proof(s)

- A trigraph G with a *skeleton* $S \subseteq G$, where S is a plane 2-connected black graph not crossed by any edges of $E(G) \setminus E(S)$.

- A BFS tree $T \subseteq G$ such that every S-face containing vertices of $V(G) \setminus V(S)$ is bounded by two T-paths $+$ edge.



- Contractions respect the BFS layers, and stick with ≤ 3 vert. per S-face and layer.

- $V(S)$ untouched by contractions, and so edges induced by $V(S)$ are black, other (except at the sink) considered red.

- Can adopt additional detailed conditions for refined proofs. . .

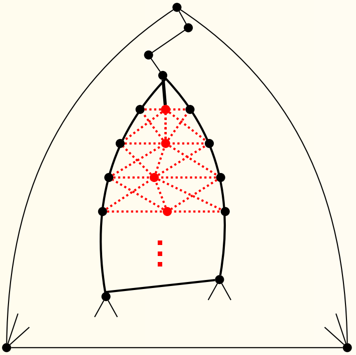
2 Basic Setup for the Proof(s)

- A trigraph G with a *skeleton* $S \subseteq G$, where S is a plane 2-connected black graph not crossed by any edges of $E(G) \setminus E(S)$.

- A BFS tree $T \subseteq G$ such that every S-face containing vertices of $V(G) \setminus V(S)$ is bounded by two T-paths $+$ edge.

- Contractions respect the BFS layers, and stick with ≤ 3 vert. per S-face and layer.

- $V(S)$ untouched by contractions, and so edges induced by $V(S)$ are black, other (except at the sink) considered red.



- Can adopt additional detailed conditions for refined proofs...

- This setup largely restricts possible red degrees $\rightarrow \leq \mathbf{11}$.

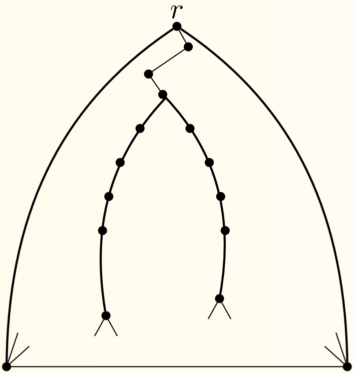
3 The Very Short Proof

- Given a simple planar graph H, extend H into a plane triangulation $H^{+} \supseteq H$ by adding vertices(!).

3 The Very Short Proof

- Given a simple planar graph H, extend H into a plane triangulation $H^+ \supseteq H$ by adding vertices(!).
- Choose a root on the outer f., and a BFS tree of H^+ from this root. *Note that all edges are only between same and successive BFS layers.*

  Start with $G = S = H^+$.



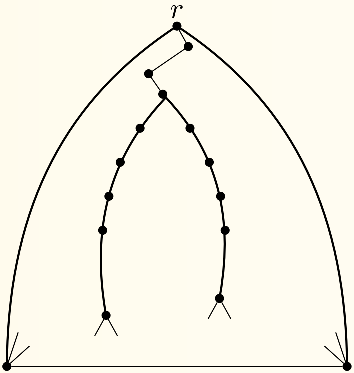
3 The Very Short Proof

- Given a simple planar graph H, extend H into a plane triangulation $H^+ \supseteq H$ by adding vertices(!).
- Choose a root on the outer f., and a BFS tree of H^+ from this root. *Note that all edges are only between same and successive BFS layers.*

  Start with $G = S = H^+$.

- Proceed by induction...

  merge two skel. faces, contract *
  (recall – we preserve BFS layers, and ≤ 3 vert. per face and layer)



3 The Very Short Proof

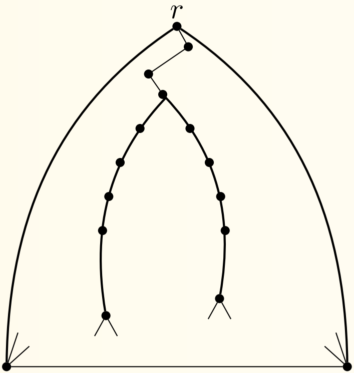
- Given a simple planar graph H, extend H into a plane triangulation $H^+ \supseteq H$ by adding vertices(!).
- Choose a root on the outer f., and a BFS tree of H^+ from this root. *Note that all edges are only between same and successive BFS layers.*

  Start with $G = S = H^+$.

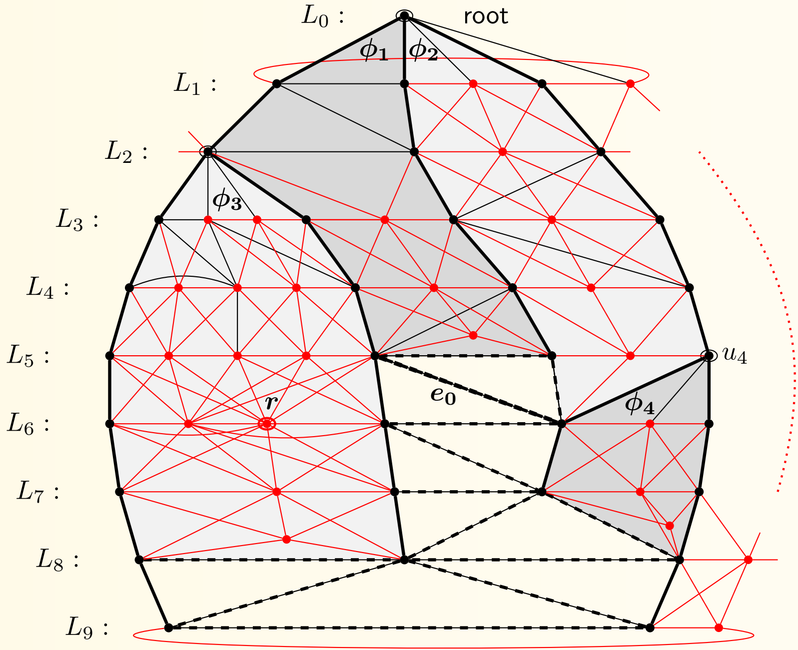
- Proceed by induction. . .

  merge two skel. faces, contract *
  (recall – we preserve BFS layers,
  and ≤ 3 vert. per face and layer)

  merge two skel. faces, contract *,
  merge two skel. faces, contract *,
  merge two skel. faces, contract *,
  . . . . . .

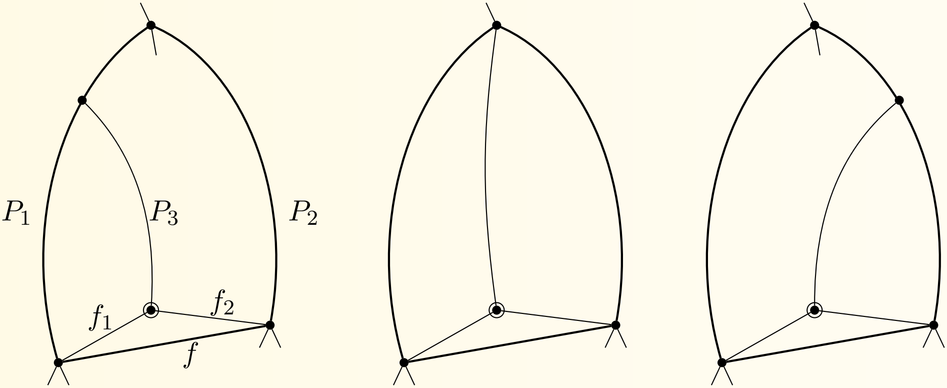


- So, how does it work?



- And the last bit – how to find the two skeleton faces to be merged?
  - Algorithmically, we'd "reverse" a natural recur. decomposition...

- And the last bit – how to find the two skeleton faces to be merged?
  - Algorithmically, we'd "reverse" a natural recur. decomposition...
  - In a theory proof, however, we just pick a minimal cycle within the current skeleton enclosing some BFS-tree leaf.



□

4 Conclusions

- While we know an upper bound of 8 (on the twin-width of planar graphs), the presented proof for ≤ 11 is way much simpler...

4 Conclusions

- While we know an upper bound of 8 (on the twin-width of planar graphs), the presented proof for ≤ 11 is way much simpler...

  A simple proof brings better understanding of planar twin-width.

4 Conclusions

- While we know an upper bound of 8 (on the twin-width of planar graphs), the presented proof for ≤ 11 is way much simpler…

  A simple proof brings better understanding of planar twin-width.

- What to try next?

4 Conclusions

- While we know an upper bound of 8 (on the twin-width of planar graphs), the presented proof for ≤ 11 is way much simpler...

  A simple proof brings better understanding of planar twin-width.

- What to try next?

  Say, a reasonable upper bound for the twin-width of *map graphs*?

4 Conclusions

- While we know an upper bound of 8 (on the twin-width of planar graphs), the presented proof for ≤ 11 is way much simpler...

  A simple proof brings better understanding of planar twin-width.

- What to try next?

  Say, a reasonable upper bound for the twin-width of *map graphs*?

  [Twin-width I] an implicit bound,
  [Bonnet–Kwon–Wood] a 7-digit bound.

4 Conclusions

- While we know an upper bound of 8 (on the twin-width of planar graphs), the presented proof for ≤ 11 is way much simpler...

  A simple proof brings better understanding of planar twin-width.

- What to try next?

  Say, a reasonable upper bound for the twin-width of *map graphs*?

  [Twin-width I] an implicit bound,
  [Bonnet–Kwon–Wood] a 7-digit bound.

- Lastly, can we use our ideas to improve the planar product structure theorem, or the planar queue number?

4 Conclusions

- While we know an upper bound of 8 (on the twin-width of planar graphs), the presented proof for ≤ 11 is way much simpler...

  A simple proof brings better understanding of planar twin-width.

- What to try next?

  Say, a reasonable upper bound for the twin-width of *map graphs*?

  [Twin-width I] an implicit bound,
  [Bonnet–Kwon–Wood] a 7-digit bound.

- Lastly, can we use our ideas to improve the planar product structure theorem, or the planar queue number?

Thank you for your attention.