On efficient solvability of graph problems parameterized by "width" (rank-width)

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Talk based on joint work with R. Ganian and J. Obdržálek.

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- Explicit comb. extensions of this concept appeared e.g. in the works [Abrahamson and Fellows, 93], [PH, 03], or [Ganian and PH, 08].

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- Informally, the classes of $\approx_{\mathcal{P},k}$ capture all information about the property $\mathcal{P}$ that can "cross" our boundary of size k
(regardless of the actual meaning of "boundary" and "join").

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- For simplicity, solution fragments φ can be "embedded" in $\mathcal{U}_k$ and $\otimes$.
- Can, e.g., count the solutions in each class of $\approx_{\mathcal{P},k}$, or keep an opt. one.

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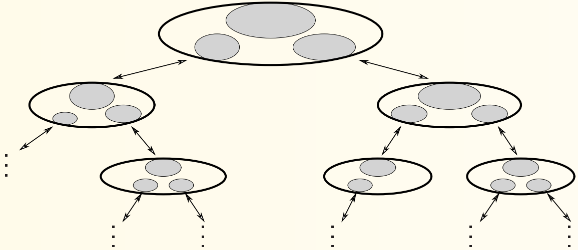
- This can be (visually) seen as...



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  $\rightarrow$ We do not need to know the equivalence classes exactly and constructively, just enough to have some (weak) estimate on them...

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- Both positive and negative examples will be given further.

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$\longrightarrow$ **A problem** – no known way how to construct an expression tree!

Rank-decomposition

- [Oum and Seymour, 03] Bringing the branch-decomposition approach to measure "complexity" of vertex subsets $X \subseteq V(G)$ via *cut-rank*:

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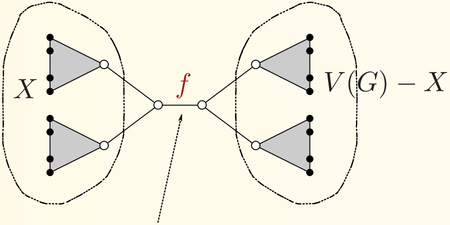

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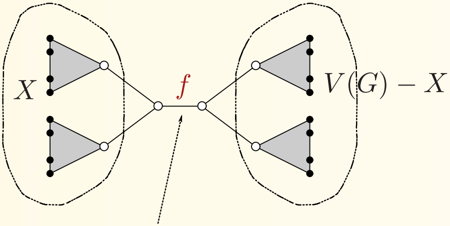

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- **Rank-width** $= \min_{\text{rank-decs. of } G} \max \{\text{width}(f) : f \text{ tree edge}\}$

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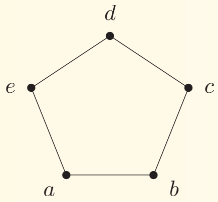
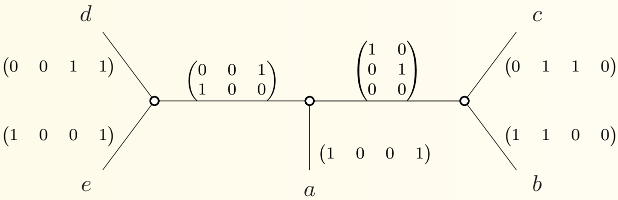
An example. Cycle C_5 and its *rank-decomposition* of width 2:





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    i.e. "odd intersection" of vertex labelings.
- Join $\to$ a *composition* operator with relabelings f_1, f_2, g;

$$(G_1, lab^1) \otimes[\boldsymbol{g} \mid \boldsymbol{f}_1, \boldsymbol{f}_2] (G_2, lab^2) = (H, lab)$$

$\Longrightarrow$ the rank-width **parse tree** [Ganian and PH, 08]:

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$\Longrightarrow$ the rank-width **parse tree** [Ganian and PH, 08]:

t-**labeling** parse tree for $G \iff$ rank-width of $G \leq \boldsymbol{t}$.

Parse trees for rank-decompositions

Unlike for tree- or clique- decompositions with obvious parse trees, what is the "boundary" and "join operation" for rank-width?

Our "boundary" includes all vertices, and "join" is just an implicit matrix.

- **Bilinear product** approach of [Courcelle and Kanté, 07]:
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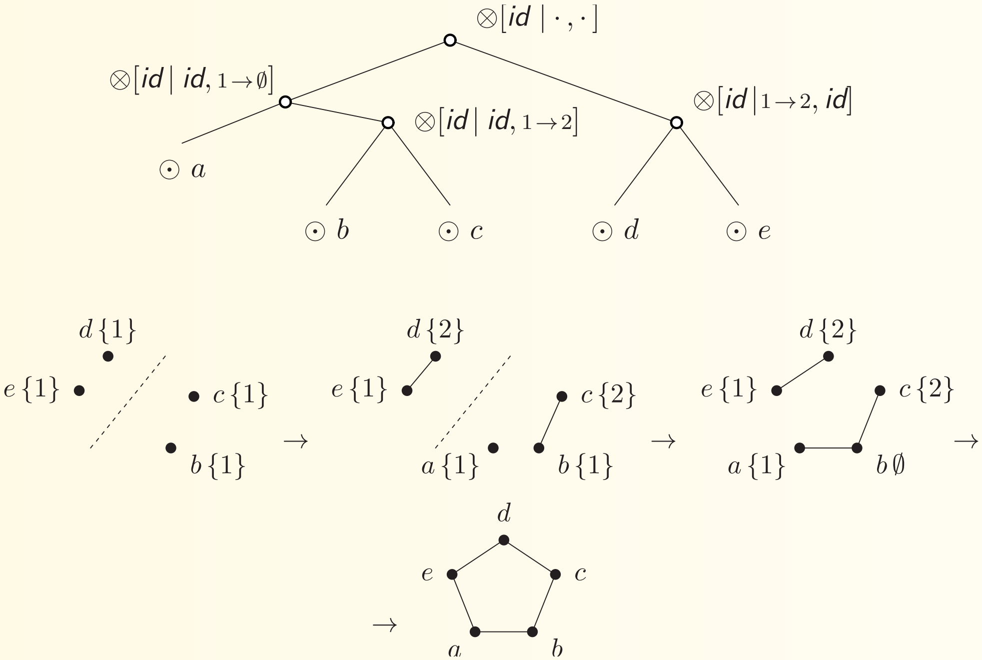
- Independently considered related notion of R_t-*join* decompositions by [Bui-Xuan, Telle, and Vatshelle, 08].

A parse tree. An example generating the cycle C_5 (of rank-width 2):



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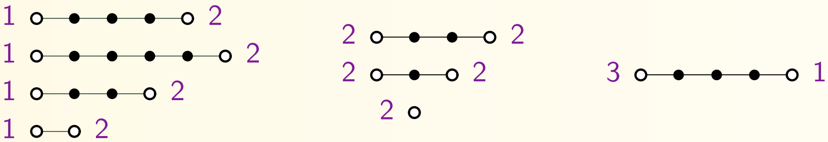

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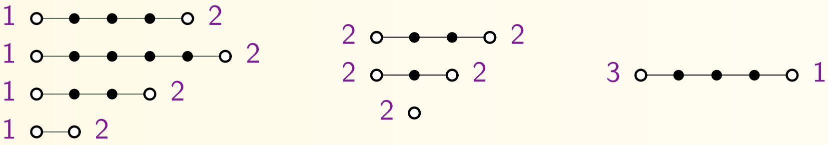

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- No, we must give also an algorithm how to “combine / process” our information on parse trees – not hard-coded this time!
- In this particular case the processing algorithm runs very smoothly. . .

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- Again, the number of classes is $O\left(n^{2^{rw^2}}\right) \prec O\left(n^{2^{cw}}\right)$,
- and there is a reasonably straightforward algorithm to "combine / process" this information on parse trees.

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– Given G and ℓ;
is there an *out-directed spanning tree* of G with $\leq \ell$ leaves?

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Actually, it looks like we face here a new situation not observed before among the known XP algorithms on bounded clique-width / rank-width graphs!

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$\Longrightarrow$ The number of equivalence classes of MinLOB is in XP.

What about an algorithm, though?

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but it does not fit into the Myhill–Nerode-like scheme!

7 Final remarks

The “naughty example” of the MinLOB problem and its XP algorithm on digraphs of bounded rank-width / clique-width raises some intrusive questions. . .

Namely:

- Is there a better refinement of the canonical equivalence of MinLOB, i.e. one that can be directly processed along a parse tree in XP time?

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THANK YOU FOR YOUR ATTENTION