On “good” and “bad” digraph width measures

Petr Hliněný*

Robert Ganian
Jan Obdržálek

Joachim Kneis
Alexander Langer
Daniel Meister
Peter Rossmanith
Somnath Sikdar

FI MU Brno

RWTH Aachen

1 How to Measure Graph "Width"

Tree-width (Robertson and Seymour) — a real success story:

- FPT algorithms for many problems, incl. all MSO_2
- structurally nice, FPT computable, *just great!*
- related to (even nicer) branch-width

1 How to Measure Graph "Width"

Tree-width (Robertson and Seymour) — a real success story:

- FPT algorithms for many problems, incl. all MSO_2
- structurally nice, FPT computable, *just great!*
- related to (even nicer) branch-width

Clique-width / rank-width (Courcelle and Olariu / Oum and Seymour)

- again, FPT or XP algorithms for many problems, incl. all MSO_1
- but not subgraph or minor-monotone

1 How to Measure Graph "Width"

Tree-width (Robertson and Seymour) — a real success story:

- FPT algorithms for many problems, incl. all MSO_2
- structurally nice, FPT computable, *just great!*
- related to (even nicer) branch-width

Clique-width / rank-width (Courcelle and Olariu / Oum and Seymour)

- again, FPT or XP algorithms for many problems, incl. all MSO_1
- but not subgraph or minor-monotone

What about directed graphs?

Directed tree-width (Johnson, Robertson, Seymour, and Thomas)

- XP algorithms for Hamiltonian path or k-path (linkage) problems
- technically difficult, not many efficient algorithms. . .

1 How to Measure Graph "Width"

Tree-width (Robertson and Seymour) — a real success story:

- FPT algorithms for many problems, incl. all MSO_2
- structurally nice, FPT computable, *just great!*
- related to (even nicer) branch-width

Clique-width / rank-width (Courcelle and Olariu / Oum and Seymour)

- again, FPT or XP algorithms for many problems, incl. all MSO_1
- but not subgraph or minor-monotone

What about directed graphs?

Directed tree-width (Johnson, Robertson, Seymour, and Thomas)

- XP algorithms for Hamiltonian path or k-path (linkage) problems
- technically difficult, not many efficient algorithms. . .

Recent additions

- an explosion of new directed measures in the past decade. . .
  giving finer resolution for better algorithmic applications ?

Directed measures: briefly (and chronologically)...

Cycle rank, —— directed path-width, dir. tree-width, *D-width, entanglement, DAG-width, Kelly-width, DFVS-number, bi-rank-width, K-width, DAG-depth*

Directed measures: briefly (and chronologically)...

Cycle rank, —— directed path-width, dir. tree-width, *D-width, entanglement, DAG-width, Kelly-width, DFVS-number, bi-rank-width, K-width, DAG-depth*

...as driven by algorithmic use:

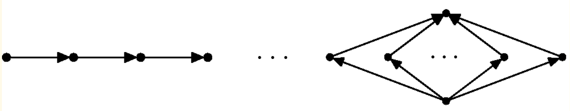
Probl. \ Param.	K-width	DAG-depth	DAG-width	Cycle-rank	DFVS-num.	DAGs	Bi-rank-width
HAM (§4.3)	**FPT**	**FPT**	XP*[a] / W[2]-hard[b]	XP*[a] / W[2]-h.[b]	XP[a] ‡	P	**XP**[c] / W[2]-h.[d]
c-PATH (§4.4)	**FPT**	**FPT**	XP*[a] ‡	XP*[a] ‡	XP[a] ‡	P[a]	**FPT**
k-PATH (§4.4)	**para-NPC**	**para-NPC**	NPC[e]	NPC[e]	NPC[e]	NPC[e]	para-NPC[f]
DIDS (§4.5)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
DISTP (§4.5)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
MAXLOB (§4.6)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
MINLOB (§4.6)	**para-NPC**	**para-NPC**	para-NPC[g]	para-NPC[g]	**para-NPC**	P[h]	*open*
c-MINLOB (§4.6)	**XP** ‡	**FPT**	XP*[g] / W[2]-hard[b]	XP*[g] / W[2]-h.[b]	XP[g] ‡	P[h]	**XP**[c] / W[2]-h.[d]
MAXDICUT (§4.7)	para-NPC[b]	para-NPC[b]	NPC[b]	NPC[b]	NPC[b]	NPC[b]	**XP**[c] / W[2]-h.[j]
c-OCN (§4.8)	**para-NPC**	**para-NPC**	NPC[k]	NPC[k]	NPC[k]	NPC[k]	**FPT**
DFVS (§4.9)	*open*	*open*	para-NPC[l]	para-NPC[l]	FPT[m]	P	**FPT**
KERNEL (§4.9)	para-NPC[n]	para-NPC[n]	para-NPC[l,n]	para-NPC[l,n]	**FPT**	P	**FPT**
ϕ-MSO$_1$MC (§4.2)	**para-NPH**	**para-NPH**	NPH	NPH	NPH	NPH	FPT[p]
ϕ-LTLMC (§4.10)	**p.-coNPH**	**p.-coNPH**	**coNPH**	**coNPH**	**coNPH**	**coNPC**	**para-coNPH**
PARITY (§4.10)	XP[q] ‡	XP[q] ‡	XP*[q] ‡	XP*[q] ‡	XP[q] ‡	P	XP[r] ‡

References [a][JRST01] [b][LKM08] [c][GHO10] [d][FGLS09] [e][EIS76] [f][GW06] [g][DGK09] [h][GRK09] [j][FGLS10] [k][CD06] [l][KO08] [m][CLL+08] [n][vL76] [p][CMR00] [q][BDHK06] [r][Obd07] .

FPT $\simeq$ runtime $O\big(f(k)\cdot n^c\big)$ XP $\simeq$ runtime $O\big(n^{f(k)}\big)$

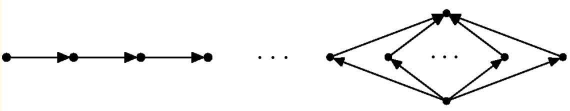
2 What are these Directed Width Measures

DAG – directed *acyclic* graph (the simplest class ???)



2 What are these Directed Width Measures

DAG – directed *acyclic* graph (the simplest class ???)

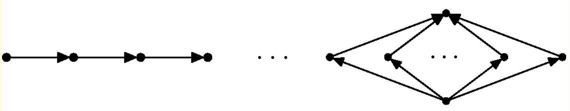


Some measures that are small on DAGs:

DAG-width – how many cops catch a *visible robber*
(no unnatural SCC restriction for the robber)

2 What are these Directed Width Measures

DAG – directed *acyclic* graph (the simplest class ???)



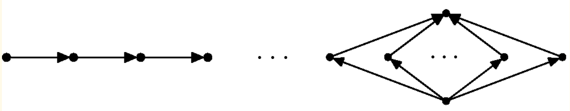
Some measures that are small on DAGs:

DAG-width – how many cops catch a *visible robber*
(no unnatural SCC restriction for the robber)

Kelly-width – how many cops catch an *invisible and lazy robber*,
or the width of a dir. elimination ordering

2 What are these Directed Width Measures

DAG – directed *acyclic* graph (the simplest class ???)



Some measures that are small on DAGs:

DAG-width – how many cops catch a *visible robber*
(no unnatural SCC restriction for the robber)

Kelly-width – how many cops catch an *invisible and lazy robber*,
or the width of a dir. elimination ordering

DFVS number – how many vertices to remove to become acyclic

2 What are these Directed Width Measures

DAG – directed *acyclic* graph (the simplest class ???)

Some measures that are small on DAGs:

DAG-width – how many cops catch a *visible robber*
(no unnatural SCC restriction for the robber)

Kelly-width – how many cops catch an *invisible and lazy robber*,
or the width of a dir. elimination ordering

DFVS number – how many vertices to remove to become acyclic

Cycle rank (60's!) – how "deep" to remove vertices to become acyclic

Some measures that are **high** on DAGs:

DAG-depth – how many cop moves are needed to catch a *visible robber*, related to the longest directed path

Some measures that are **high** on DAGs:

DAG-depth – how many cop moves are needed to catch a *visible robber*, related to the longest directed path

K-width – how many distinct paths between a pair of vertices

Some measures that are high on DAGs:

DAG-depth – how many cop moves are needed to catch a *visible robber*, related to the longest directed path

K-width – how many distinct paths between a pair of vertices

and slightly different sort...

Clique-width – same def. for undirected and directed:

Minimum number of *labels* to build the graph using

- create a (labeled) vertex,
- make disjoint union,
- relabel all i's to j,
- and add all arcs from label i to j.

Some measures that are high on DAGs:

DAG-depth – how many cop moves are needed to catch a *visible robber*, related to the longest directed path

K-width – how many distinct paths between a pair of vertices

and slightly different sort...

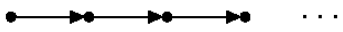
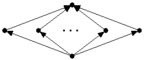
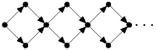
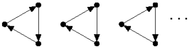
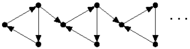
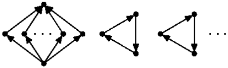
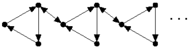
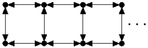
Clique-width – same def. for undirected and directed:

Minimum number of *labels* to build the graph using

- create a (labeled) vertex,
- make disjoint union,
- relabel all i's to j,
- and add all arcs from label i to j.

Bi-rank-width (Kanté) – related to clique-width / rank-width; i.e. the branch-width of the *bi-cutrank* function on the vertex set.

How these measures compare

Graph family	DAG-depth	K-width	DFVS-number	cycle-rank	DAG-width
	∞	1	0	0	1
	3	∞	0	0	1
	∞	∞	0	0	1
	3	1	∞	1	2
	∞	1	∞	1	2
	3	∞	∞	1	2
	∞	1	∞	∞	3
	∞	∞	∞	∞	3

3 Their Structural Properties

Very good: DAG-width, Kelly-width, DAG-depth

- having nice cops-and-robber *game characterizations*
- monotone under taking subgraphs and some restricted form of *arc contractions*

3 Their Structural Properties

Very good: DAG-width, Kelly-width, DAG-depth

- having nice cops-and-robber *game characterizations*
- monotone under taking subgraphs and some restricted form of *arc contractions*

Good: directed tree-width, DFVS number, cycle rank, K-width

- no game chars., but still monotone under taking subgraphs

3 Their Structural Properties

Very good: DAG-width, Kelly-width, DAG-depth

- having nice cops-and-robber *game characterizations*
- monotone under taking subgraphs and some restricted form of *arc contractions*

Good: directed tree-width, DFVS number, cycle rank, K-width

- no game chars., but still monotone under taking subgraphs

---

and Bad: clique-width, bi-rank-width

- subgraphs can have much higher width,
  e.g. the complete graph (bidirected) has small width while its subgraphs are complex
- still, not so bad since related to so called *vertex minors*

4 and Algorithmic Usefulness

Probl. \ Param.	K-width	DAG-depth	DAG-width	Cycle-rank	DFVS-num.	DAGs	Bi-rank-width
HAM (§4.3)	**FPT**	**FPT**	XP*[a] / W[2]-hard[b]	XP*[a] / W[2]-h.[b]	XP[a] ‡	P	**XP**[c] / W[2]-h.[d]
c-Path (§4.4)	**FPT**	**FPT**	XP*[a] ‡	XP*[a] ‡	XP[a] ‡	P[a]	**FPT**
k-Path (§4.4)	**para-NPC**	**para-NPC**	NPC[e]	NPC[e]	NPC[e]	NPC[e]	para-NPC[f]
DiDS (§4.5)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
DiSTP (§4.5)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
MaxLOB (§4.6)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
MinLOB (§4.6)	**para-NPC**	**para-NPC**	para-NPC[g]	para-NPC[g]	**para-NPC**	P[h]	*open*
c-MinLOB (§4.6)	**XP** ‡	**FPT**	XP*[g] / W[2]-hard[b]	XP*[g] / W[2]-h.[b]	XP[g] ‡	P[h]	**XP**[c] / W[2]-h.[d]
MaxDiCut (§4.7)	para-NPC[b]	para-NPC[b]	NPC[b]	NPC[b]	NPC[b]	NPC[b]	**XP**[c] / W[2]-h.[j]
c-OCN (§4.8)	**para-NPC**	**para-NPC**	NPC[k]	NPC[k]	NPC[k]	NPC[k]	**FPT**
DFVS (§4.9)	*open*	*open*	para-NPC[l]	para-NPC[l]	FPT[m]	P	**FPT**
Kernel (§4.9)	para-NPC[n]	para-NPC[n]	para-NPC[l,n]	para-NPC[l,n]	**FPT**	P	**FPT**
ϕ-MSO$_1$Mc (§4.2)	**para-NPH**	**para-NPH**	NPH	NPH	NPH	NPH	FPT[p]
ϕ-LTLMc (§4.10)	**p.-coNPH**	**p.-coNPH**	**coNPH**	**coNPH**	**coNPH**	**coNPC**	**para-coNPH**
Parity (§4.10)	XP[q] ‡	XP[q] ‡	XP*[q] ‡	XP*[q] ‡	XP[q] ‡	P	XP[r] ‡

References [a][JRST01] [b][LKM08] [c][GHO10] [d][FGLS09] [e][EIS76] [f][GW06] [g][DGK09] [h][GRK09] [j][FGLS10] [k][CD06] [l][KO08] [m][CLL+08] [n][vL76] [p][CMR00] [q][BDHK06] [r][Obd07] .

FPT $\simeq$ runtime $O\big(f(k)\cdot n^c\big)$ XP $\simeq$ runtime $O\big(n^{f(k)}\big)$

NPC $\simeq$ lik. no efficient alg. at all W[i]-hard $\simeq$ lik. no better than XP alg.

Conclusions from the Table. . .

Very good: clique-width, bi-rank-width

- all MSO_1 properties have FPT algorithms
- and many other problems have (at least) XP algorithms

Conclusions from the Table. . .

Very good: clique-width, bi-rank-width

- all MSO_1 properties have FPT algorithms
- and many other problems have (at least) XP algorithms

Moderate: DAGs

- but this is not a measure, just a special case!

Conclusions from the Table...

Very good: clique-width, bi-rank-width

- all MSO_1 properties have FPT algorithms
- and many other problems have (at least) XP algorithms

Moderate: DAGs

- but this is not a measure, just a special case!

---

and Bad: all the other measures!

Conclusions from the Table...

Very good: clique-width, bi-rank-width

- all MSO_1 properties have FPT algorithms
- and many other problems have (at least) XP algorithms

Moderate: DAGs

- but this is not a measure, just a special case!

---

and Bad: all the other measures!

- classical digraph problems like dominating set, Steiner tree, max- / min-LOB (outbranching), oriented colouring, etc. are still NP-hard for the measures

Conclusions from the Table. . .

Very good: clique-width, bi-rank-width

- all MSO_1 properties have FPT algorithms
- and many other problems have (at least) XP algorithms

Moderate: DAGs

- but this is not a measure, just a special case!

---

and Bad: all the other measures!

- classical digraph problems like dominating set, Steiner tree, max- / min-LOB (outbranching), oriented colouring, etc. are still NP-hard for the measures
- positive algorithmic results seem rather incidental,
  e.g. Hamiltonian path and related, or some particular algorithms parametrized by the DFVS number

5 Can we do better?

The contrast: So far we have got no directed measure that is structurally nice and algorithmically useful at the same time!

5 Can we do better?

The contrast: So far we have got no directed measure that is structurally nice and algorithmically useful at the same time!

The Question:

What "*structural*" and *algorithmically useful* measures of digraphs can we get?

5 Can we do better?

The contrast: So far we have got no directed measure that is structurally nice and algorithmically useful at the same time!

The Question:

What "*structural*" and *algorithmically useful* measures of digraphs can we get?

Say, the number of vertices? No. . .

5 Can we do better?

The contrast: So far we have got no directed measure that is structurally nice and algorithmically useful at the same time!

The Question:

What "*structural*" and *algorithmically useful* measures of digraphs can we get?

Say, the number of vertices? No...

Ordinary tree-width of the underlying undirected graph!

- efficiently solves almost all usual problems, incl. MSO_2
- and has quite nice structural properties, just ignore the directions

5 Can we do better?

The contrast: So far we have got no directed measure that is structurally nice and algorithmically useful at the same time!

The Question:

What "*structural*" and *algorithmically useful* measures of digraphs can we get?

Say, the number of vertices? No...

Ordinary tree-width of the underlying undirected graph!

- efficiently solves almost all usual problems, incl. MSO_2
- and has quite nice structural properties, just ignore the directions

OK, but we want a directed measure that is

NOT tree-width bounding!

The Question, II:

Can we have an *algorithmically useful* measure of digraphs that is not tree-width bounding and monotone on subgraphs (i.e. "*structural*")?

The Question, II:

Can we have an *algorithmically useful* measure of digraphs that is not tree-width bounding and monotone on subgraphs (i.e. "*structural*")?

This "crazy subdivision" measure works well:

- $\mathbf{0}$ if every two vertices of deg > 2 are "very far" apart, $|V|$ otherwise
- again, efficiently solves almost all usual problems, incl. MSO_2

The Question, II:

Can we have an *algorithmically useful* measure of digraphs that is not tree-width bounding and monotone on subgraphs (i.e. "*structural*")?

This "crazy subdivision" measure works well:

- $\mathbf{0}$ if every two vertices of deg > 2 are "very far" apart, $|V|$ otherwise
- again, efficiently solves almost all usual problems, incl. MSO_2

NO, we really do not want a measure like this one, right?

The Question, II:

Can we have an *algorithmically useful* measure of digraphs that is not tree-width bounding and monotone on subgraphs (i.e. "*structural*")?

This "crazy subdivision" measure works well:

- $\mathbf{0}$ if every two vertices of deg > 2 are "very far" apart, $|V|$ otherwise
- again, efficiently solves almost all usual problems, incl. MSO_2

NO, we really do not want a measure like this one, right?

The Question, II':

What about add. monotonicity under *butterfly contractions* (minors)?

The Question, II:

Can we have an *algorithmically useful* measure of digraphs that is not tree-width bounding and monotone on subgraphs (i.e. "*structural*")?

This "crazy subdivision" measure works well:

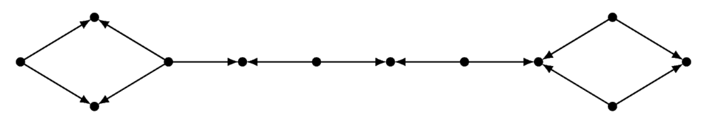
- $\mathbf{0}$ if every two vertices of deg > 2 are "very far" apart, $|V|$ otherwise
- again, efficiently solves almost all usual problems, incl. MSO_2

NO, we really do not want a measure like this one, right?

The Question, II':

What about add. monotonicity under *butterfly contractions* (minors)?

NO, this does not help to dismiss the "crazy" measure either...



The Question, III:

So, what definition of a *directed minor* shall we consider when describing the property of being "structurally nice"?

The Question, III:

So, what definition of a *directed minor* shall we consider when describing the property of being "structurally nice"?

- contractions that do not create any new directed paths (cf. the butterfly minors) are not helpful in our context

The Question, III:

So, what definition of a *directed minor* shall we consider when describing the property of being "structurally nice"?

- contractions that do not create any new directed paths (cf. the butterfly minors) are not helpful in our context
  — we need to contract any induced "long path" (even not directed)!

The Question, III:

So, what definition of a *directed minor* shall we consider when describing the property of being "structurally nice"?

- contractions that do not create any new directed paths (cf. the butterfly minors) are not helpful in our context
  — we need to contract any induced "long path" (even not directed)!
- instead, we choose to define *directed topological minors* as follows:
  - let V_3 be the subset of vertices with > 2 neighbours;
  - arc $\vec{a}$ is *2-contractible* if
    - not both ends of $\vec{a}$ are in V_3, and
    - no new dir. path between vert. of V_3 after contraction of $\vec{a}$

The Question, III:

So, what definition of a *directed minor* shall we consider when describing the property of being "structurally nice"?

- contractions that do not create any new directed paths (cf. the butterfly minors) are not helpful in our context
  — we need to contract any induced "long path" (even not directed)!
- instead, we choose to define *directed topological minors* as follows:
  - let V_3 be the subset of vertices with > 2 neighbours;
  - arc $\vec{a}$ is *2-contractible* if
    - not both ends of $\vec{a}$ are in V_3, and
    - no new dir. path between vert. of V_3 after contraction of $\vec{a}$

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,

The Question, III:

So, what definition of a *directed minor* shall we consider when describing the property of being "structurally nice"?

- contractions that do not create any new directed paths (cf. the butterfly minors) are not helpful in our context
  — we need to contract any induced "long path" (even not directed)!
- instead, we choose to define *directed topological minors* as follows:
  - let V_3 be the subset of vertices with > 2 neighbours;
  - arc $\vec{a}$ is *2-contractible* if
    - not both ends of $\vec{a}$ are in V_3, and
    - no new dir. path between vert. of V_3 after contraction of $\vec{a}$

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and

The Question, III:

So, what definition of a *directed minor* shall we consider when describing the property of being "structurally nice"?

- contractions that do not create any new directed paths (cf. the butterfly minors) are not helpful in our context
  — we need to contract any induced "long path" (even not directed)!
- instead, we choose to define *directed topological minors* as follows:
  - let V_3 be the subset of vertices with > 2 neighbours;
  - arc $\vec{a}$ is *2-contractible* if
    - not both ends of $\vec{a}$ are in V_3, and
    - no new dir. path between vert. of V_3 after contraction of $\vec{a}$

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

6 No, we cannot do better – our Answer

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

Powerfulness - why undirected MSO_1?

6 No, we cannot do better – our Answer

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

Powerfulness - why undirected MSO_1?

- A useful width measure should not only incidentally solve a few problems, but a whole rich class (a *framework*).

6 No, we cannot do better – our Answer

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

Powerfulness - why undirected MSO_1?

- A useful width measure should not only incidentally solve a few problems, but a whole rich class (a *framework*).
- Say, we would like to solve problems in a **logic-based framework**, then:

6 No, we cannot do better – our Answer

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

Powerfulness - why undirected MSO_1?

- A useful width measure should not only incidentally solve a few problems, but a whole rich class (a *framework*).
- Say, we would like to solve problems in a **logic-based framework**, then:
  - ability to test the presence of an arc (u, v), plus

6 No, we cannot do better – our Answer

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

Powerfulness - why undirected MSO_1?

- A useful width measure should not only incidentally solve a few problems, but a whole rich class (a *framework*).
- Say, we would like to solve problems in a **logic-based framework**, then:
  - ability to test the presence of an arc (u, v), plus
  - the language of (at least) MSO to capture global properties

6 No, we cannot do better – our Answer

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

Powerfulness - why undirected MSO_1?

- A useful width measure should not only incidentally solve a few problems, but a whole rich class (a *framework*).
- Say, we would like to solve problems in a **logic-based framework**, then:
  - ability to test the presence of an arc (u, v), plus
  - the language of (at least) MSO to capture global properties
  - $\Longrightarrow$ **undirected MSO_1 is the least common denominator!**

And why efficiently orientable?

I.e., for every undirected G, one can efficiently orient (in XP time) the edges of G such that the width is (approximately) optimal over all orientations of G.

And why efficiently orientable?

I.e., for every undirected G, one can efficiently orient (in XP time) the edges of G such that the width is (approximately) optimal over all orientations of G.

- Traditional directed measures are efficiently orientable.

And why efficiently orientable?

I.e., for every undirected G, one can efficiently orient (in XP time) the edges of G such that the width is (approximately) optimal over all orientations of G.

- Traditional directed measures are efficiently orientable.
- Giving up this condition, we could encode computationally excessive information (NP-compl. oracle) in the orientation of edges.

And why efficiently orientable?

I.e., for every undirected G, one can efficiently orient (in XP time) the edges of G such that the width is (approximately) optimal over all orientations of G.

- Traditional directed measures are efficiently orientable.
- Giving up this condition, we could encode computationally excessive information (NP-compl. oracle) in the orientation of edges.
- Such exc. encoding can even be preserved on dir. topol. minors!

And why efficiently orientable?

I.e., for every undirected G, one can efficiently orient (in XP time) the edges of G such that the width is (approximately) optimal over all orientations of G.

- Traditional directed measures are efficiently orientable.
- Giving up this condition, we could encode computationally excessive information (NP-compl. oracle) in the orientation of edges.
- Such exc. encoding can even be preserved on dir. topol. minors!

3-colouring encoding example — low width if the arcs *encode a 3-colouring*:

- arcs directed from lower to higher colour

And why efficiently orientable?

I.e., for every undirected G, one can efficiently orient (in XP time) the edges of G such that the width is (approximately) optimal over all orientations of G.

- Traditional directed measures are efficiently orientable.
- Giving up this condition, we could encode computationally excessive information (NP-compl. oracle) in the orientation of edges.
- Such exc. encoding can even be preserved on dir. topol. minors!

3-colouring encoding example — low width if the arcs *encode a 3-colouring*:

  - arcs directed from lower to higher colour
  - condition: having any dir. path with ends of deg. > 2, the start is a source or the end is a sink
    (and this cond. is closed under dir. topol. minors)

And why efficiently orientable?

I.e., for every undirected G, one can efficiently orient (in XP time) the edges of G such that the width is (approximately) optimal over all orientations of G.

- Traditional directed measures are efficiently orientable.
- Giving up this condition, we could encode computationally excessive information (NP-compl. oracle) in the orientation of edges.
- Such exc. encoding can even be preserved on dir. topol. minors!

3-colouring encoding example — low width if the arcs *encode a 3-colouring*:

- arcs directed from lower to higher colour
- condition: having any dir. path with ends of deg. > 2, the start is a source or the end is a sink
  (and this cond. is closed under dir. topol. minors)
- excessive info. – even knowing a graph is 3-colourable, there is no efficient way to find a colouring (this measure is **cheating**!)

7 The Conclusion, again

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

7 The Conclusion, again

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

– not tree-width bounding,
– monotone under taking directed topological minors,
– *efficiently orientable* (approx. in XP), and
– algorithmically *powerful* (undirected MSO_1 in XP).

- As argued above, these assumptions are all natural,

7 The Conclusion, again

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

– not tree-width bounding,
– monotone under taking directed topological minors,
– *efficiently orientable* (approx. in XP), and
– algorithmically *powerful* (undirected MSO_1 in XP).

- As argued above, these assumptions are all natural,

  and there is no solution fulfilling all of them!

7 The Conclusion, again

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

- As argued above, these assumptions are all natural,

  and there is no solution fulfilling all of them!

- So, which of the assumptions should be given up?

7 The Conclusion, again

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

- As argued above, these assumptions are all natural,

  and there is no solution fulfilling all of them!

- So, which of the assumptions should be given up?

  Our point of view is *algorithmic*, and so the only possibility here to give up is the structural condition!

7 The Conclusion, again

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

– not tree-width bounding,
– monotone under taking directed topological minors,
– *efficiently orientable* (approx. in XP), and
– algorithmically *powerful* (undirected MSO_1 in XP).

- As argued above, these assumptions are all natural,

  and there is no solution fulfilling all of them!

- So, which of the assumptions should be given up?

  Our point of view is *algorithmic*, and so the only possibility here to give up is the structural condition!

- Hence, for algorithmically useful directed measures, we can not require nice structural properties at the same time, and thus. . .

7 The Conclusion, again

Theorem. Unless P=NP, there is **NO directed width measure** s.t.

- not tree-width bounding,
- monotone under taking directed topological minors,
- *efficiently orientable* (approx. in XP), and
- algorithmically *powerful* (undirected MSO_1 in XP).

- As argued above, these assumptions are all natural,

  and there is no solution fulfilling all of them!

- So, which of the assumptions should be given up?

  Our point of view is *algorithmic*, and so the only possibility here to give up is the structural condition!

- Hence, for algorithmically useful directed measures, we can not require nice structural properties at the same time, and thus. . .

- **Bi-rank-width is a really good dir. measure** – the best we (can) have?

THANK YOU FOR YOUR ATTENTION