How “Good” Digraph Width Measures Do / Can We Have?

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1 How to Measure Graph "Width"

Tree-width (Robertson and Seymour) — a real success story:

- FPT algorithms for many problems, incl. all MSO_2
- structurally nice, FPT computable, *just great!*
- related to (even nicer) branch-width

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Recent additions

- an explosion of new directed measures in the past decade. . .
  giving finer resolution for better algorithmic applications ?

Directed measures: briefly (and chronologically). . .

Cycle rank, —— directed path-width, dir. tree-width, *D-width, entanglement, DAG-width, Kelly-width, DFVS-number, bi-rank-width, K-width, DAG-depth*

Directed measures: briefly (and chronologically)...

Cycle rank, —— directed path-width, dir. tree-width, *D-width, entanglement, DAG-width, Kelly-width, DFVS-number, bi-rank-width, K-width, DAG-depth*

...as driven by algorithmic use:

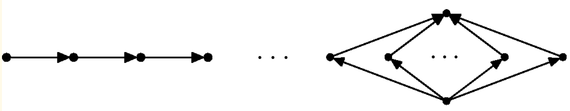
Probl. \ Param.	K-width	DAG-depth	DAG-width	Cycle-rank	DFVS-num.	DAGs	Bi-rank-width
HAM (§4.3)	**FPT**	**FPT**	XP*[a] / W[2]-hard[b]	XP*[a] / W[2]-h.[b]	XP[a] ‡	P	**XP**[c] / W[2]-h.[d]
c-Path (§4.4)	**FPT**	**FPT**	XP*[a] ‡	XP*[a] ‡	XP[a] ‡	P[a]	**FPT**
k-Path (§4.4)	**para-NPC**	**para-NPC**	NPC[e]	NPC[e]	NPC[e]	NPC[e]	para-NPC[f]
DiDS (§4.5)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
DiSTP (§4.5)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
MaxLOB (§4.6)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
MinLOB (§4.6)	**para-NPC**	**para-NPC**	para-NPC[g]	para-NPC[g]	**para-NPC**	P[h]	*open*
c-MinLOB (§4.6)	**XP** ‡	**FPT**	XP*[g] / W[2]-hard[b]	XP*[g] / W[2]-h.[b]	XP[g] ‡	P[h]	**XP**[c] / W[2]-h.[d]
MaxDiCut (§4.7)	para-NPC[b]	para-NPC[b]	NPC[b]	NPC[b]	NPC[b]	NPC[b]	**XP**[c] / W[2]-h.[j]
c-OCN (§4.8)	**para-NPC**	**para-NPC**	NPC[k]	NPC[k]	NPC[k]	NPC[k]	**FPT**
DFVS (§4.9)	*open*	*open*	para-NPC[l]	para-NPC[l]	FPT[m]	P	**FPT**
Kernel (§4.9)	para-NPC[n]	para-NPC[n]	para-NPC[l,n]	para-NPC[l,n]	**FPT**	P	**FPT**
ϕ-MSO$_1$MC (§4.2)	**para-NPH**	**para-NPH**	NPH	NPH	NPH	NPH	FPT[p]
ϕ-LTLMC (§4.10)	**p.-coNPH**	**p.-coNPH**	**coNPH**	**coNPH**	**coNPH**	**coNPC**	**para-coNPH**
Parity (§4.10)	XP[q] ‡	XP[q] ‡	XP*[q] ‡	XP*[q] ‡	XP[q] ‡	P	XP[r] ‡

References [a][JRST01] [b][LKM08] [c][GHO10] [d][FGLS09] [e][EIS76] [f][GW06] [g][DGK09] [h][GRK09] [j][FGLS10] [k][CD06] [l][KO08] [m][CLL$^+$08] [n][vL76] [p][CMR00] [q][BDHK06] [r][Obd07] .

FPT $\simeq$ runtime $O\big(f(k) \cdot n^c\big)$ XP $\simeq$ runtime $O\big(n^{f(k)}\big)$

2 What are these Directed Width Measures

DAG – directed *acyclic* graph (the simplest class ???)



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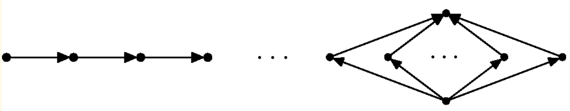

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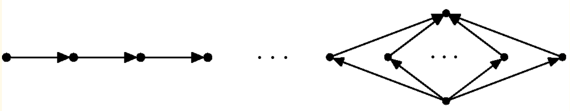

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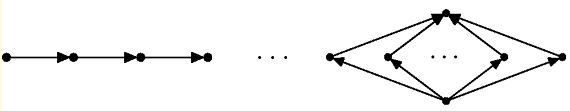

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Cycle rank (60's!) – how "deep" to remove vertices to become acyclic

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Minimum number of *labels* to build the graph using

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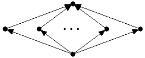
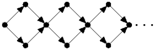
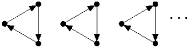
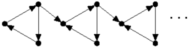
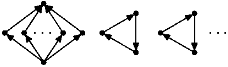
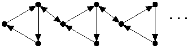
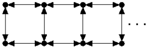
Bi-rank-width (Kanté) – related to clique-width / rank-width; i.e. the branch-width of the *bi-cutrank* function on the vertex set.

How these measures compare

Graph family	DAG-depth	K-width	DFVS-number	cycle-rank	DAG-width
	∞	1	0	0	1
	3	∞	0	0	1
	∞	∞	0	0	1
	3	1	∞	1	2
	∞	1	∞	1	2
	3	∞	∞	1	2
	∞	1	∞	∞	3
	∞	∞	∞	∞	3

3 Their Structural Properties

Very good: DAG-width, Kelly-width, DAG-depth

- having nice cops-and-robber *game characterizations*
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and Bad: clique-width, bi-rank-width

- subgraphs can have much higher width,
  e.g. the complete graph (bidirected) has small width while its subgraphs are complex
- still, not so bad since related to so called *vertex minors*

4 and Algorithmic Usefulness

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c-PATH (§4.4)	**FPT**	**FPT**	XP*[a] ‡	XP*[a] ‡	XP[a] ‡	P[a]	**FPT**
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DIDS (§4.5)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
DISTP (§4.5)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
MAXLOB (§4.6)	**para-NPC**	**para-NPC**	**NPC**	**NPC**	**NPC**	**NPC**	**FPT**
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c-MINLOB (§4.6)	**XP** ‡	**FPT**	XP*[g] / W[2]-hard[b]	XP*[g] / W[2]-h.[b]	XP[g] ‡	P[h]	**XP**[c] / W[2]-h.[d]
MAXDICUT (§4.7)	para-NPC[b]	para-NPC[b]	NPC[b]	NPC[b]	NPC[b]	NPC[b]	**XP**[c] / W[2]-h.[j]
c-OCN (§4.8)	**para-NPC**	**para-NPC**	NPC[k]	NPC[k]	NPC[k]	NPC[k]	**FPT**
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FPT $\simeq$ runtime $O\big(f(k)\cdot n^c\big)$ XP $\simeq$ runtime $O\big(n^{f(k)}\big)$

NPC $\simeq$ lik. no efficient alg. at all W[i]-hard $\simeq$ lik. no better than XP alg.

Conclusions from the Table. . .

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- classical digraph problems like dominating set, Steiner tree, max- / min-LOB (outbranching), oriented colouring, etc. are still NP-hard for the measures
- positive algorithmic results seem rather incidental, e.g. Hamiltonian path and related, or some particular algorithms parametrized by the DFVS number

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OK, but we want a directed measure that is

NOT tree-width bounding!

The Question, II:

Can we have an *algorithmically useful* measure of digraphs that is not tree-width bounding and monotone on subgraphs (i.e. "*structural*")?

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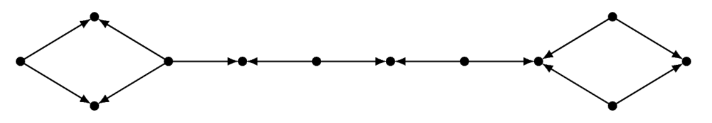
NO, this does not help to dismiss the "crazy" measure either. . .



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  - let V_3 be the subset of vertices with > 2 neighbours;
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Powerfulness - why undirected MSO_1?

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  - $\Longrightarrow$ **undirected MSO_1 is the least common denominator!**

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  - excessive info. – even knowing a graph is 3-colourable, there is no efficient way to find a colouring (this measure is **cheating**!)

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- **Bi-rank-width is a really good dir. measure** – the best we (can) have?

THANK YOU FOR YOUR ATTENTION