Testing FO properties of dense structures

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Based on joint works with J. Gajarský[†], D. Lokshtanov[**], J. Obdržálek[*], S. Ordyniak[‡], M.S. Ramanujan[‡], and S. Saurabh[**].

[*] MU Brno, [**] Univ. Bergen, [†] TU Berlin, [‡] TU Wien

First-order logic

Definition (FO)

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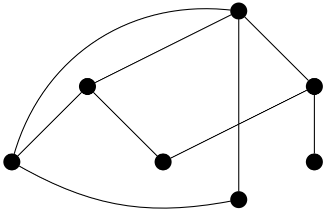

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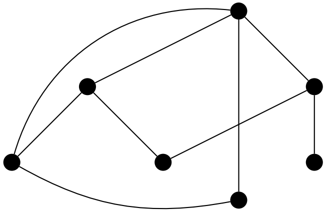

"There is no isolated vertex."

First-order logic

- $\psi(x, y) \equiv \forall z : z = x \lor z = y \lor \text{edge}(x, z) \lor \text{edge}(y, z)$?

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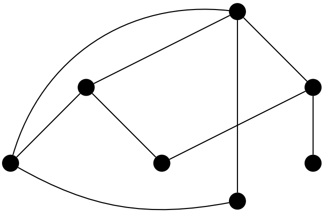

"$\{x, y\}$ is a dominating set."

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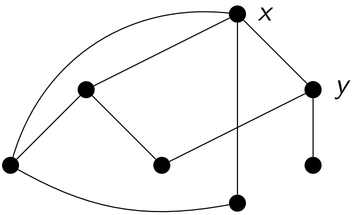

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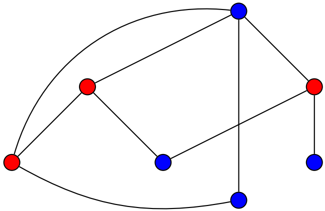
Coloured FO logic

- $\phi \equiv \forall x, y : \big[(\mathsf{red}(x) \wedge \mathsf{red}(y)) \rightarrow \neg\mathsf{edge}(x, y)\big] \wedge \big[(\mathsf{blue}(x) \wedge \mathsf{blue}(y)) \rightarrow \neg\mathsf{edge}(x, y)\big]$?



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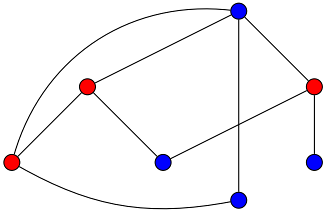

"Given is a proper 2-colouring?"

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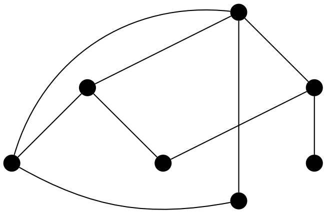

"The input graph has a dominating set of size ≤ 2."

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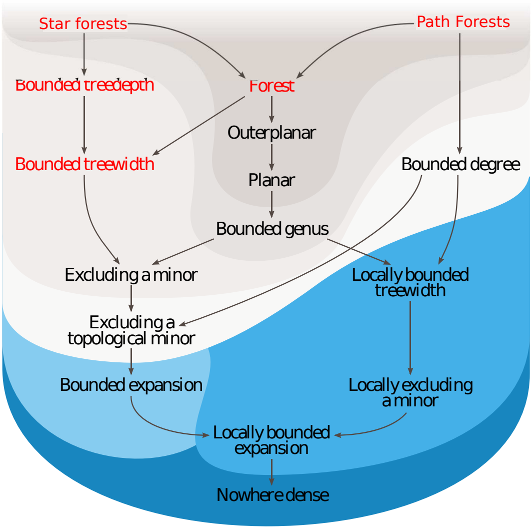
  Answer:

  - In general – **no**, W-hard (cf. indep. or dominating set).
  - For restricted graph classes – **yes**.

FO Model Checking of Sparse Graphs



- (Courcelle – MSO)

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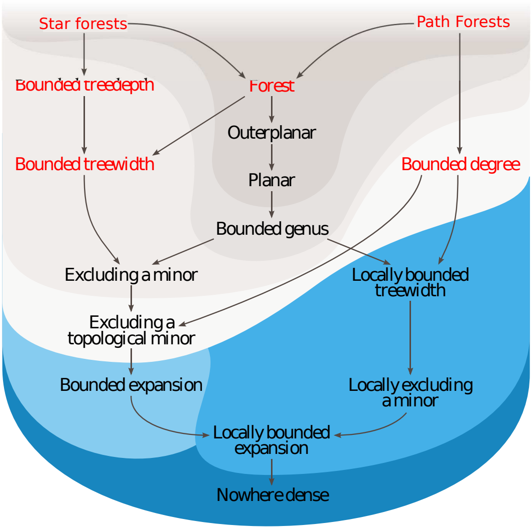

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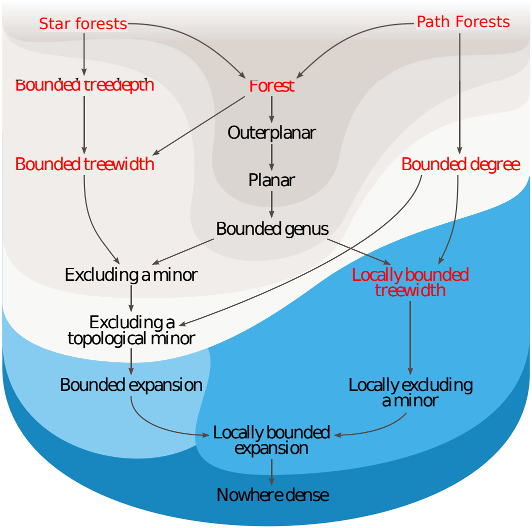

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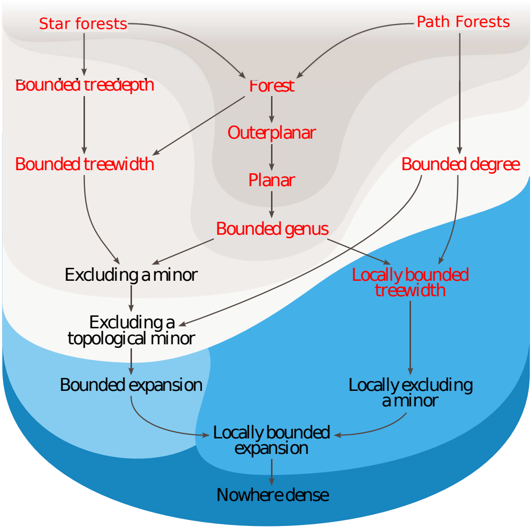

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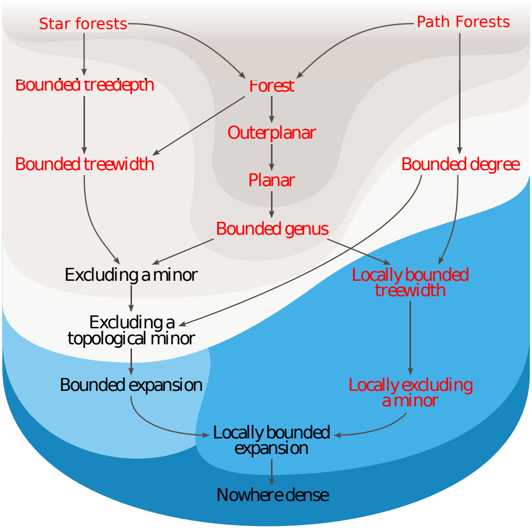

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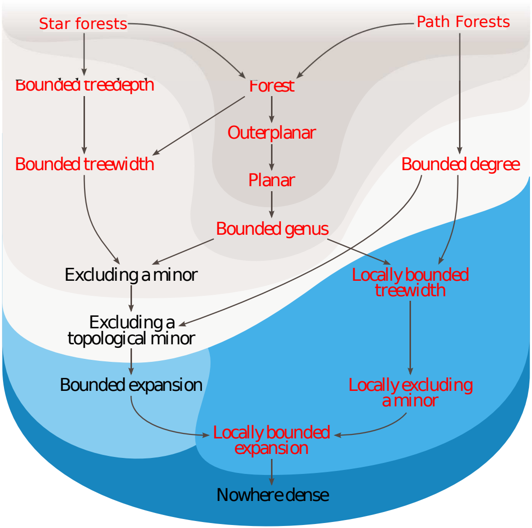

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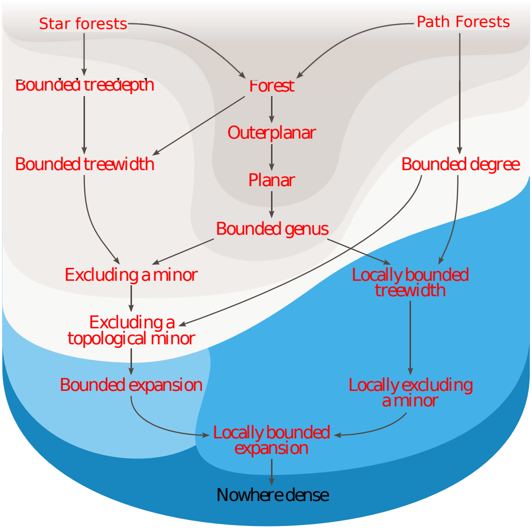

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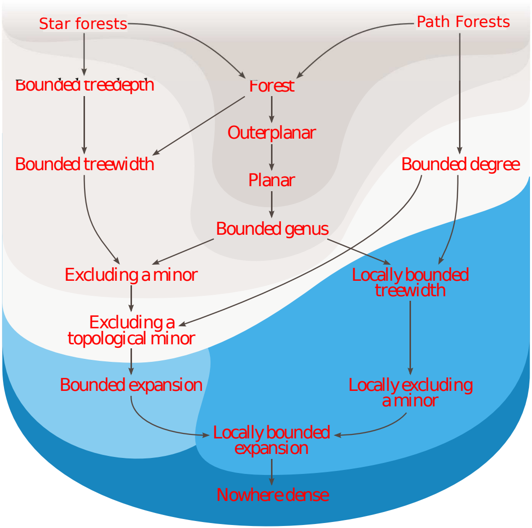

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FO on sparse graphs

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Theorem (Gaifman locality theorem)

To evaluate a formula ϕ on G it is enough to:

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Neighbourhood in relational structures – in the Gaifman graph which has a clique for every tuple of each relation.

Beyond sparsity?

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1. Consider other (dense) graph classes, e.g.
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How to continue? Two basic options:

1. Consider other (dense) graph classes, e.g.
   - *L*-interval graphs [Ganian et al., 2013]
2. Consider other kinds of structures like
   - posets [2014], lattices, finite groups, ...?

Interval graphs

Definition (INT)

Representation: a set $\mathcal{I}$ of intervals on the real line.

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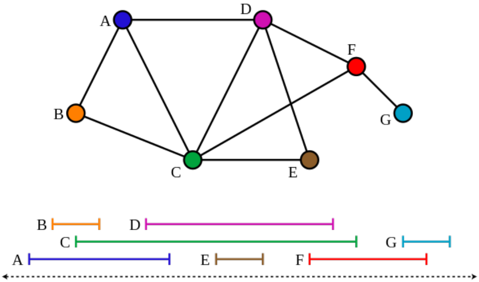

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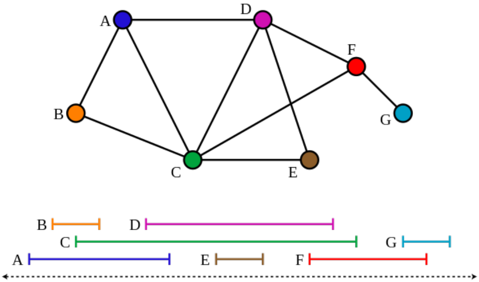

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INT & all interval lengths are from a *fixed (finite)* set $L \subseteq \mathbb{R}^+$.

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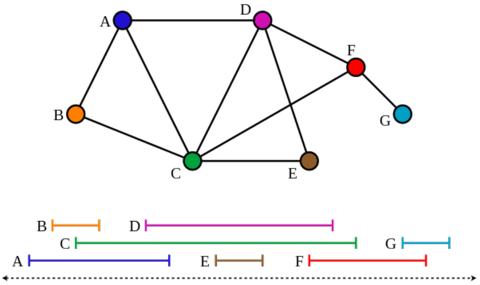

Why this particular case?

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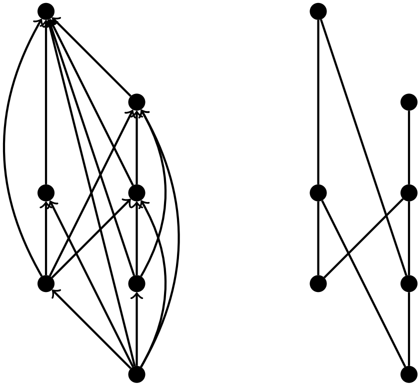
FO model checking on L-INT graphs is

1. *FPT for any* finite *set* $L \subseteq \mathbb{R}^+$, *and*
2. *W-hard for any* $\varepsilon > 0$ *and* $L = (1, 1 + \varepsilon)$.

Partially ordered sets – Posets

Definition (Poset)

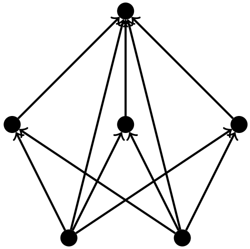
Poset $\mathcal{P} = (P, \leq)$ is a set P together with relation $\leq$ which is reflexive, antisymmetric and transitive.



FO logic on posets

(Posets are typically dense directed graphs.)

- $\phi \equiv \exists x \forall y : (x \geq y)$



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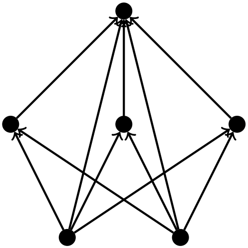
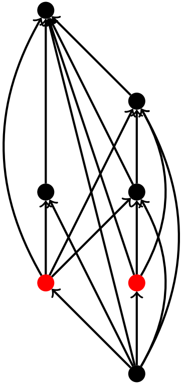


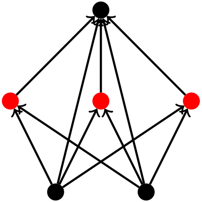
"The poset has a maximum element."

Poset width

Definition

Width of a poset = the size of its largest antichain.





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- the (2014) main result – about posets of bounded **width**.

FO model checking on posets of bounded width

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POSET ∃-FO MODEL CHECKING *solvable in time* $f(\phi) \cdot n^{g(w)}$

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Theorem (Gajarský, PH, Obdržálek and Ordyniak; ISAAC 2014)

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Towards full FO on posets

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Is it possible to solve **full** *FO model checking problem in time $f(\phi) \cdot n^{g(w)}$ or $f(\phi, w) \cdot n^{O(1)}$?*

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Theorem (Gajarský, PH, Lokshtanov, Obdržálek, Ordyniak, . Ramanujan, Saurabh; FOCS 2015)

(Full) FO model checking on posets of width at most w is solvable in time $f(\phi, w) \cdot n^2$.

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Tools we use:

- Hintikka games
- New version of locality

Hintikka games

Played on the structure and formula by two players:

- Existential player (Verifier) – plays $\vee$ and $\exists$
- Universal player (Falsifier) – plays $\wedge$ and $\forall$

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Theorem (folklore)

Given a structure $\mathcal{S}$ and a formula ϕ, the existential player has a winning strategy in the game $\mathcal{G}(\mathcal{S}, \phi)$ if, and only if, $\mathcal{S} \models \phi$.

Main idea

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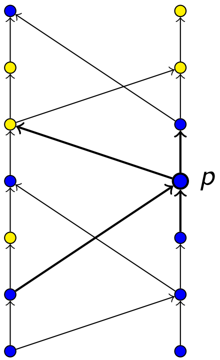
Graph D is built inductively by the structure of ϕ...

The construction of D_0

(bounded-width) $\mathcal{P} \to D_0 \to D_1 \to \cdots \to D_s$



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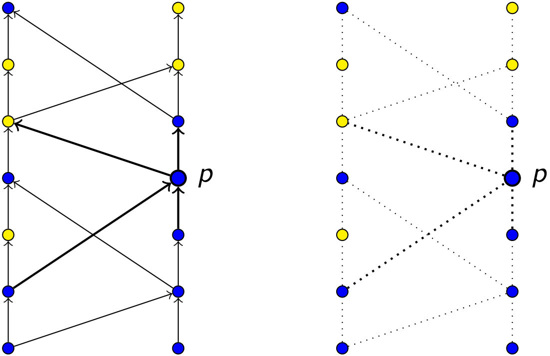

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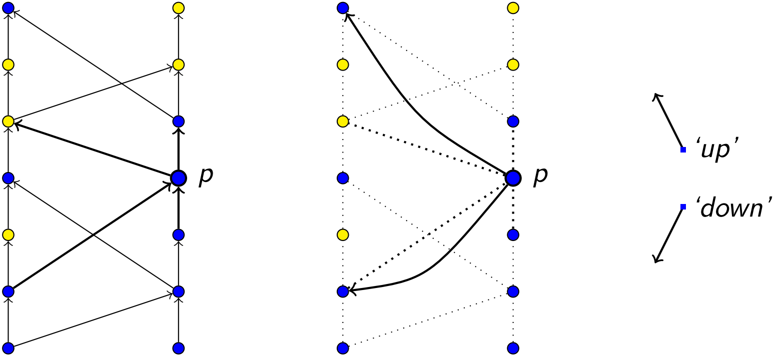

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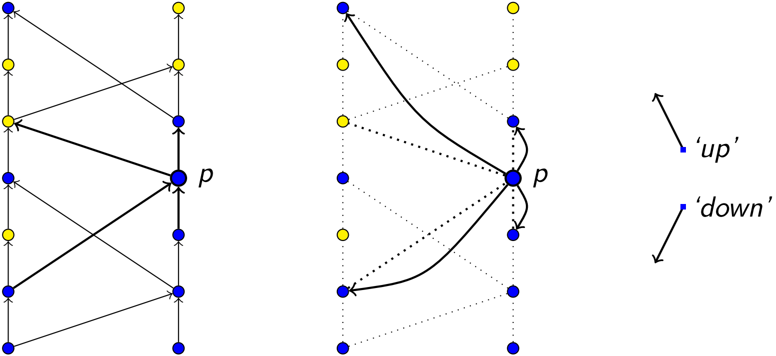

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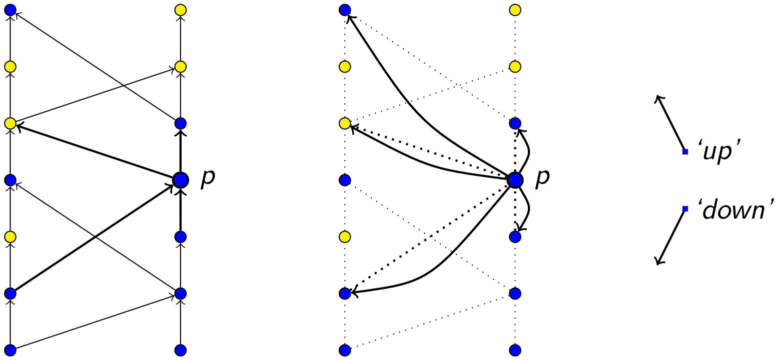

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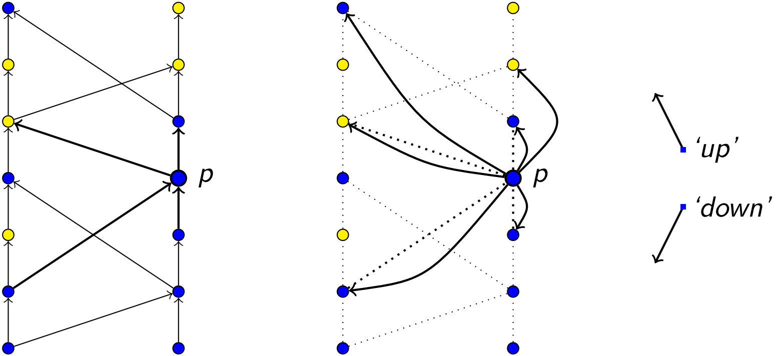

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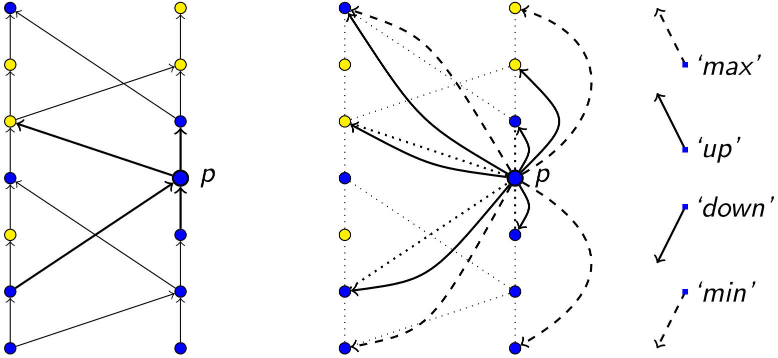

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In D_{s-1} (think of D_0) we have:

- finitely many chains
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$\Rightarrow$ there are finitely many non-isomorphic k-outneighbourhoods

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Two vertices have the same **type** if they have isomorphic k-outneighbourhoods.

Iterating the construction

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To get D_s from D_{s-1} we:

- Compute the type of each vertex in D_{s-1}.
- Use it as colour in the next round to construct D_s in the same way D_0 was constructed.

Back to the problem

Recall...

Theorem

The existential player has a winning strategy in the Hintikka game $\mathcal{G}(\mathcal{P}, \phi)$ if, and only if, $\mathcal{P} \models \phi$.

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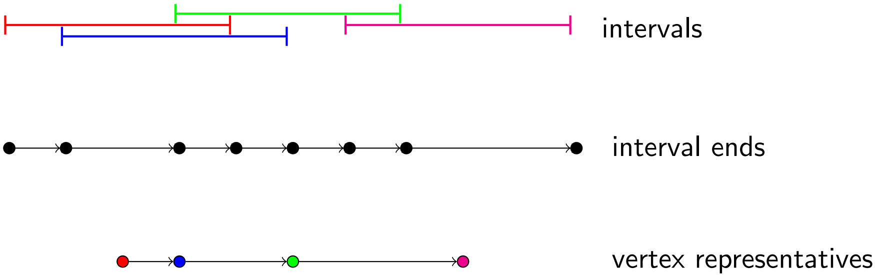
Finally, the local game is played on posets of **bounded size** and the algorithm follows.

Revisiting interval graphs

For simplicity, we restrict to unit-interval graphs; i.e., $L = \{1\}$.



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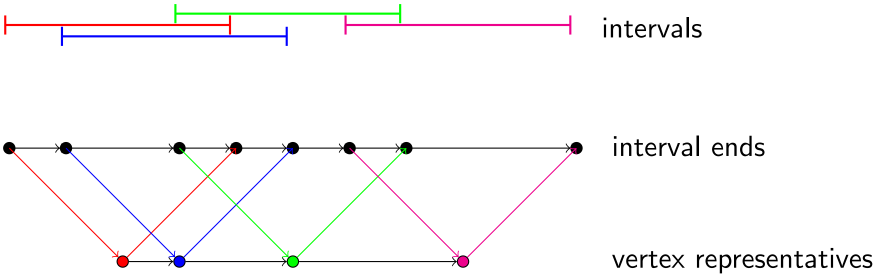

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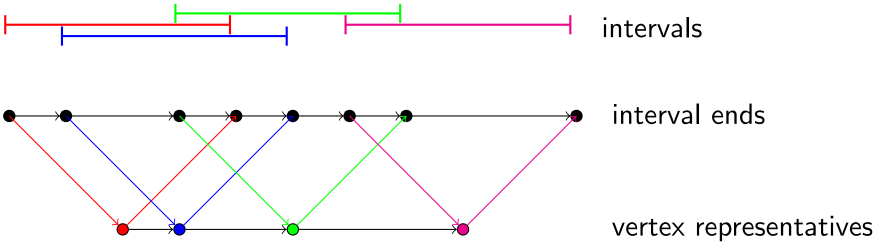

This can be generalized to bounded-nesting interval graphs:

Theorem

FO model checking on interval graphs, such that no w intervals form a "nesting chain", is solvable in time $f(\phi, w) \cdot n^2$.

Another approach – Interpretations

For a graph G and an FO formula $\psi(x, y)$, define a graph H:

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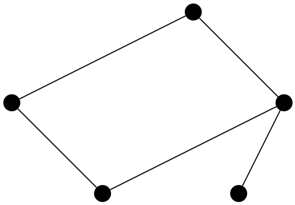

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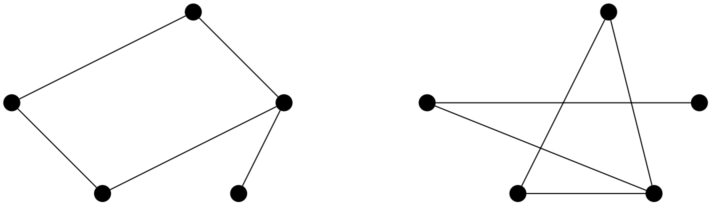

"The complement of a graph."

Another approach – Interpretations

Consider classes $\mathcal{K}$, $\mathcal{M}$ of relational structures, and a construction:

over $\mathcal{K}$:		over $\mathcal{M}$:
FO ϕ	$\xrightarrow{I}$	FO ϕ^I (edge$(x,y) \rightsquigarrow \psi(x,y)$)
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Then I is called a simple FO interpretation between $\mathcal{K}$, $\mathcal{M}$.

Lemma

$$G \models \phi \ \ \textit{iff} \ \ H \models \phi^I$$

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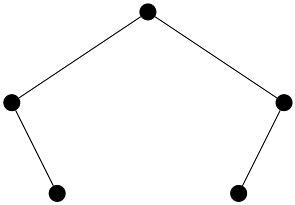
- $\psi(x, y) \equiv x \neq y \wedge \left[\text{edge}(x, y) \vee \exists z(\text{edge}(x, z) \wedge \text{edge}(z, y))\right]$



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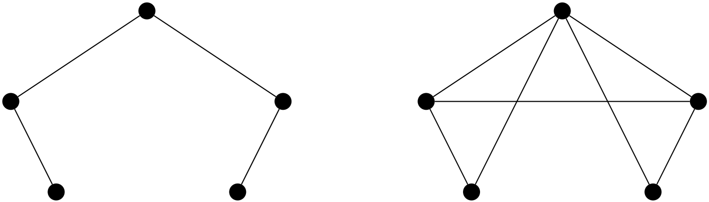

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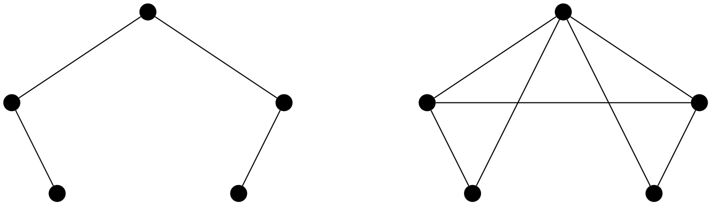

"The square of a graph."

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$$\text{FO } \phi \xrightarrow{\;I\;} \text{FO } \phi^I \quad (\,\text{edge}(x,y) \rightsquigarrow \psi(x,y)\,)$$

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Unfortunately, **not**. . .

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Theorem (Motwani and Sudan, 1994)

The problem to compute a "square root" of a graph is NP-hard.

FO interpretation in classes of bounded degree

Theorem (Gajarský, PH, Lokshtanov, Obdržálek, Ramanujan; . LICS 2016)

Let $\mathcal{D}$ be a graph class having an FO interpretation into a class of graphs of bounded degree. Then there exist an FPT algorithm for FO model checking on $\mathcal{D}$.

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In fact, the following is proved:

Theorem

Let $\mathcal{D}$ be a graph class having an FO interpretation I into a class of graphs of bounded degree. Then there exists an FO interpretation J such that, for given $G \in \mathcal{D}$, we can efficiently compute H of bounded degree such that $H^J \cong G$.

Note that one cannot require $J = I$...

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"Bounded degree away from bounded neighbourhood diversity."

Intermezzo: Neighbourhood diversity

- For a graph G, we say that two vertices $u, v \in V(G)$ are *twins* if $N(u)\backslash v = N(v)\backslash u$ (true twins).
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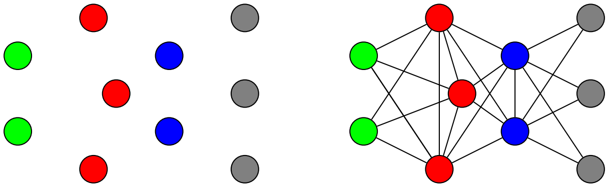
FO interpretable in edgeless k-coloured graphs:

$$\psi(x, y) \equiv (Green(x) \wedge Red(y)) \vee (Red(x) \wedge Blue(y)) \vee (Blue(x) \wedge Gray(y)) \vee (Red(x) \wedge Red(y)) \vee (Blue(x) \wedge Blue(y))$$



Near-k-twin relation

We would like to formally capture the words

"bounded degree away from bounded neighbourhood diversity."

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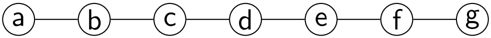
The near-k-twin relation is **not** always an equivalence!



- $k = 1$: $a \sim c$ and $e \sim g$, this is an equivalence
- $k = 2$: $a \sim c \sim e$ but $a \not\sim e$, **not** an equivalence
- $k = 4$: one equivalence class

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Graph class $\mathcal{K}$ is *near-uniform* if there exist k_0 and m such that for every $G \in \mathcal{K}$ there exists $k \leq k_0$ s.t. the near-k-twin relation is an equivalence on $V(G)$ with at most m classes.

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Summarizing the algorithm

Input: $H \in \mathcal{D}$ (where $\mathcal{D}$ is (k_0, m)-near-uniform), FO formula ϕ
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4. Use some known efficient model checking algorithm for graphs of bounded degree to determine whether $G \models \phi'$.

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Thank you for your attention!