Algorithms for embedded graphs

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(based on work by/with several people)

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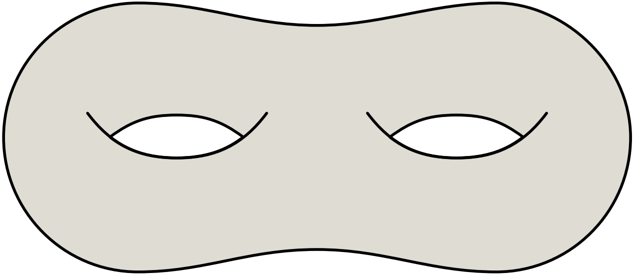
Outline

- Topology and graphs on surfaces
- Algorithmic problems in embedded graphs
- Sample of techniques
- FPTness of crossing number
- Stretch

Surfaces

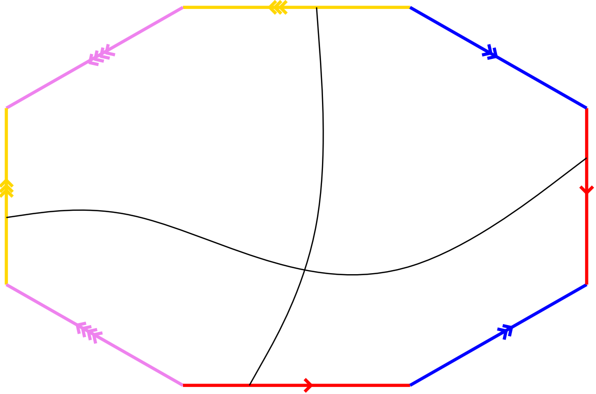
A (topological) surface is something that, locally, looks like $\mathbb{R}^2$



We restrict ourselves to compact, orientable surfaces:
each is homeomorphic to a sphere with g handles attached to it
We say the genus of the graph is g

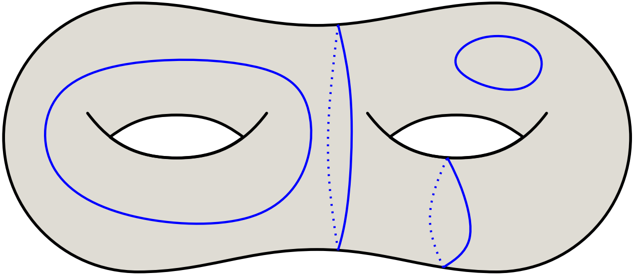
Surfaces – Polygonal schema

A double torus ($g = 2$) using a polygonal schema



Curves on Surfaces

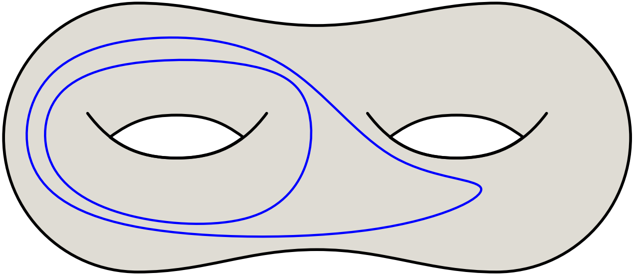
A closed curve is a continuous mapping $\alpha : \mathbb{S}^1 \to$ surface



It is *simple* if it has no self-intersections (injective)

Topological Concepts

- α, β closed curves
- α, β are homotopic if α can be continuously deformed to β
- deformation within the surface



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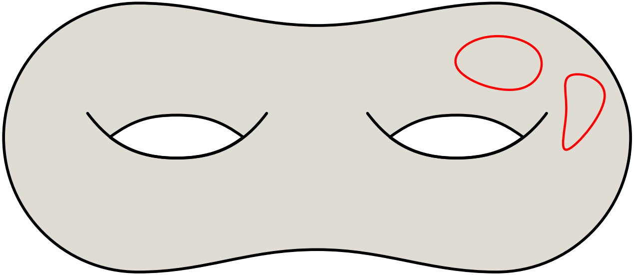

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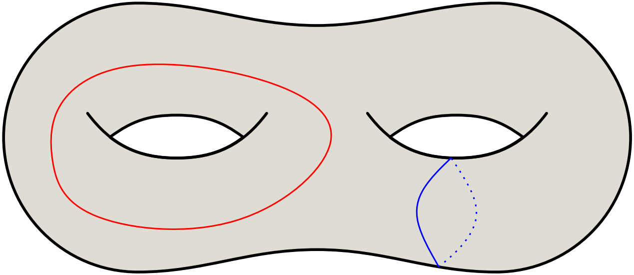
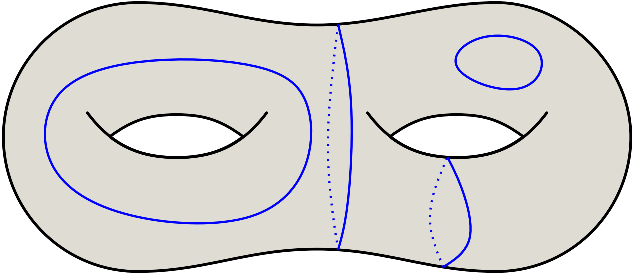


Contractible

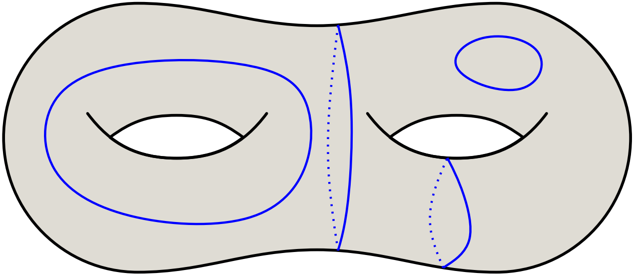
- α simple closed curve
- α is *contractible* if it is homotopic to a constant mapping



Theorem: α contractible and simple $\Rightarrow$ α bounds a disk

Separating

- α closed curve
- α is *separating* if removing its image disconnects the surface
- related to $\mathbb{Z}_2$-homology

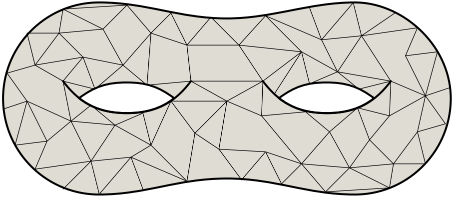


Theorem: Non-separating $\Rightarrow$ Non-contractible

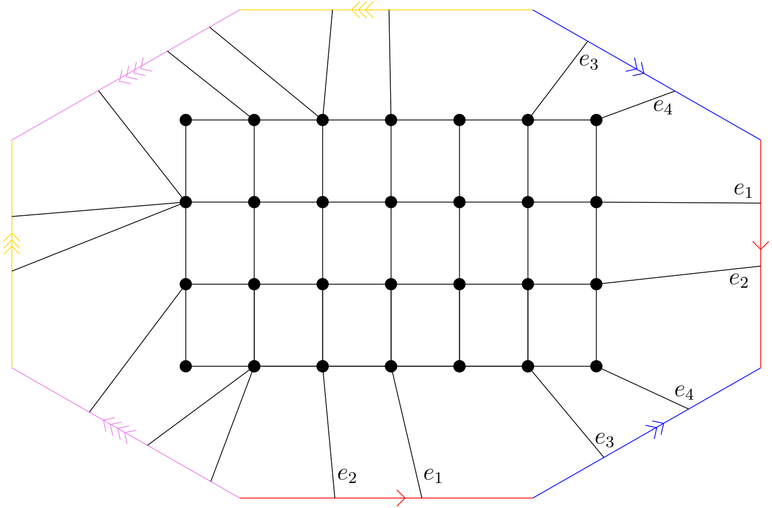
Embedded Graphs

G is embedded in a surface if:

- each vertex $u \in V(G)$ assigned to a distinct point u
- each edge uv assigned to a simple curve connecting u to v
- interior of edges disjoint from other edges and $V(G)$
- each face is a topological disk (2-cell embedding)



Embedded Graphs – Polygonal Schema



Representations of Embedded Graphs

- rotation system: for each vertex, the circular ordering of its outgoing edges as DCL.
- coordinate-less DCEL:
  - halfedges
  - vertices
  - faces
  - adjacency relations between them
- flags or gem representation
- ...

The surface is implicit in the representation of the graph.

Surgery should be doable efficiently.

Embeddable vs Embedded

- planar graph: can be embedded in the plane
- plane graph: a particular embedding
- an embedding can be obtained from the abstract planar graph in linear time

Embeddable vs Embedded

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- g-graph: can be embedded in g-surface
- embedded g-graph: a particular embedding
- NP-complete: is G a g-graph? [Thomassen '89]
- The problem is fpt wrt genus g [Mohar '99]
  - "simpler" algorithm by Kawarabayahi, Mohar and Reed 2008
  - $2^{O(g)}n$ time for any *fixed* surface

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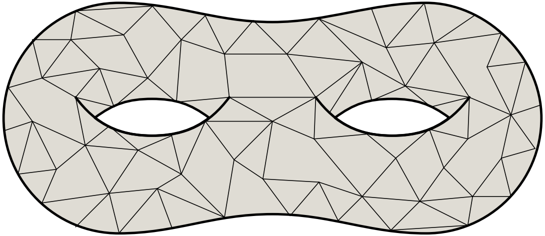
Our scenario

Input: an embedded graph G with (abstract) edge-lengths
Cycles/closed walks in G are closed curves in the surface



Actors: algorithms, topology, and the metric d_G
$n \equiv$ complexity of the input graph: $|E(G)|$
The case $g \ll n$ or even $g = O(1)$ is relevant

Algorithmic problems

Input: embedded graph with edge-lengths

- find a shortest non-contractible/non-separating cycle
- find a shortest contractible cycle/*walk*
- given α, find the shortest cycle homotopic/homologous to α
- find a cycle shortest in its homotopy/homology class
- max *s*-*t* flow
- find a shortest planarizing set
- a 'good' representation of distances in embedded graphs

Shortest non-contractible cycle

- most popular and traditional problem
- subroutine for other problems
  - crossing number: does a graph have crossing number $\leq k$?
  - approximation algorithms for TSP in embedded graphs or near-planar graphs [Demaine, Hajiaghayi, Mohar '07]
  - numerical analysis for Hodge decomposition
- overlap with analysis of meshes arising from scanned data
  - removal of topological noise [Wood et al. '04]
  - identification of handles and tunnels [Dey et al. '08]

Find a shortest non-contractible cycle

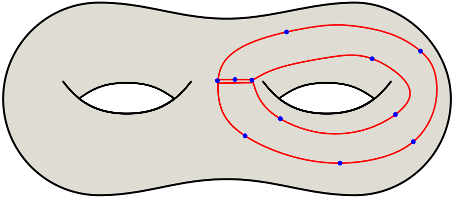
Race

- C. Thomassen – $O(n^3 \log n)$ '90
- J. Erickson and S. Har-Peled – $O(n^2 \log n)$ '02
- S. Cabello and B. Mohar – $O(g^{O(g)} n^{3/2} \log n)$ '05
- S. Cabello – $O(g^{O(g)} n^{4/3})$ '06
- M. Kutz – $O(g^{O(g)} n \log n)$ '06
- S. Cabello, E. Chambers and J. Erickson $O(g^2 n \log n)$ '12
- S. Cabello, E. Colin de Verdiere and F. Lazarus $O(gnk)$ '12

All them also work for non-separating, but no metatheorem

Shortest contractible curve

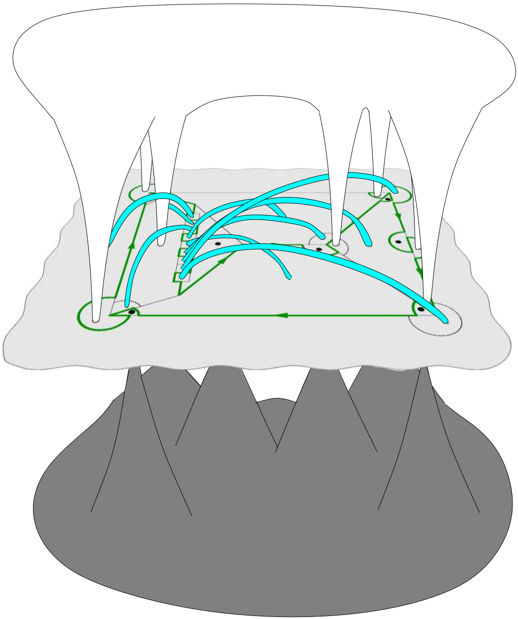
- contractible closed walk
  - does not need to be a circuit
  - not difficult to solve in polynomial time
  - $O(n \log n)$ [Cabello, DeVos, Erickson, Mohar '10]
    using [Lkacki, Sankowski '11]



- contractible cycle without repeated vertices
  - $O(n^2 \log n)$ [Cabello '10]
  - shortest cycle in planar graph with forbidden pairs

Separating cycles

- does it exists *any* separating cycle without repeated vertices?
  - NP-hard [Cabello, Colin de Verdière, and Lazarus '10]
  - reduction from Hamiltonian cycle in 3-regular planar graphs



Summary of results (up to date?)

	Cycle	Closed walk
Contractible	$O(n^2 \log n)$	$O(n \log n)$
Separating	NP-hard	???, FPT wrt g
Non-contractible	$O(\min\{g^2, n\} n \log n)$	$\leftarrow$ same
Non-separating	$O(\min\{g^2, n\} n \log n)$	$\leftarrow$ same
Tight	$\uparrow$ same	$O(n \log n)$
Splitting	NP-hard	NP-hard, FPT wrt g
Prescribed homotopy	???	nice polynomial
Prescribed homology	NP-hard, FPT wrt g	$\leftarrow$ same

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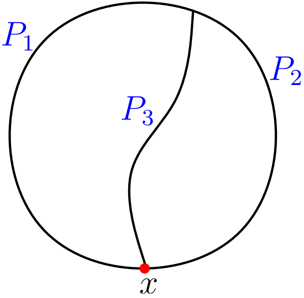
Unique shortest paths via Isolation Lemma

- unique shortest path between any two vertices
- probabilistically enforced using Isolation Lemma:
  - perturb each edge-length $\ell(e)$ by $k_e \cdot \varepsilon$, where $k_e \in \{1, \ldots, |E|^2\}$ at random
  - each shortest path is unique whp
  - more efficient than lexicographic comparison
- simpler arguments

3-path condition

P_1, P_2, P_3 three paths from $x \in V(G)$ to a common endpoint

loops $P_1 + P_3$ and $P_2 + P_3$
contractible
$\Downarrow$
loop $P_1 + P_2$ contractible



- shortest non-contractible loop from x made of two shortest paths
- if T_x shortest path tree from x, only loops $loop(T_x, e)$ are candidates
- there are $|E(G)| - (n-1)$ candidate loops

3-path condition

Set L_x of loops from x satisfies 3-path condition if:

for any three paths P_1, P_2, P_3 from x to a common endpoint, if $P_1 + P_3$ and $P_2 + P_3$ are in L_x, then $P_1 + P_2$ is in L_x

- $L_x \sim$ zeros in some sense
- contractible loops
- loops with even number of edges
- shortest loop from x outside L_x (non-zero) is made of two shortest paths and an edge
- if membership in L_x is testable in polynomial time, finding shortest loop outside L_x solvable in polynomial time

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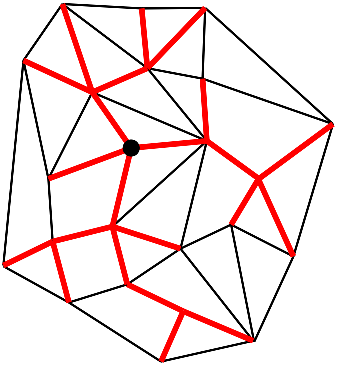
- if membership in L_x is testable in polynomial time, finding shortest loop outside L_x solvable in polynomial time
- iterate over $x \in V(G)$ for global shortest

Tree-cotree partition - Planar

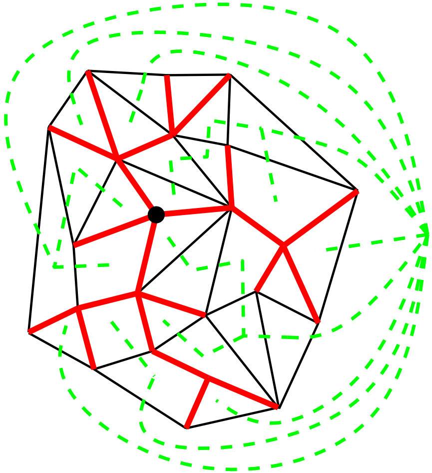
G planar. T a spanning tree



Tree-cotree partition - Planar

G planar. T a spanning tree

$G^* - E(T)^*$ is a spanning tree of the dual graph G^*

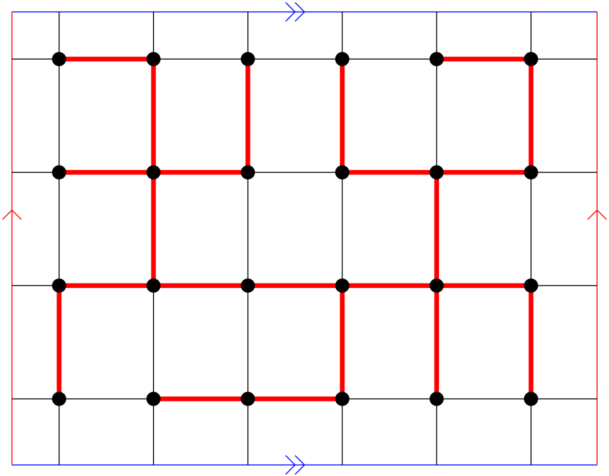


Tree-cotree partition - General

G embedded graph.
T a spanning tree of G
$C \subset E(G)$ cotree: C^* spanning tree of G^* disjoint from $E(T)^*$
X edges not in T or C. $X = \{e \in E(G) \mid e \notin E(T) \cup E(C)\}$

- (T, C, X) is a tree-cotree partition
- X has $2g$ edges (orientable) or g edges (non-orientable)
- (C^*, T^*, X^*) a tree-cotree partition of G^*
- for any $e \in X$, the cycle in $T + e$ is non-separating

Tree-cotree partition - Example



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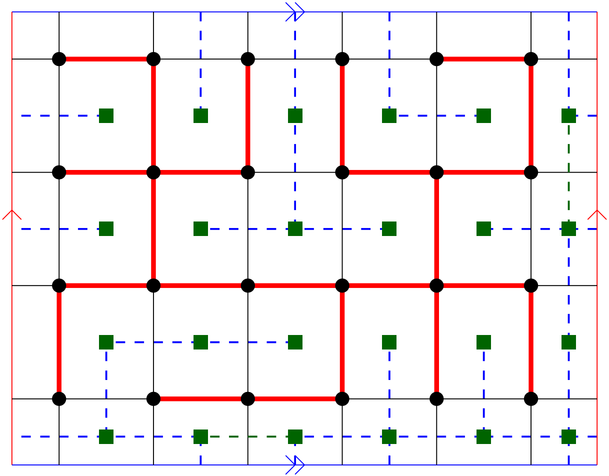

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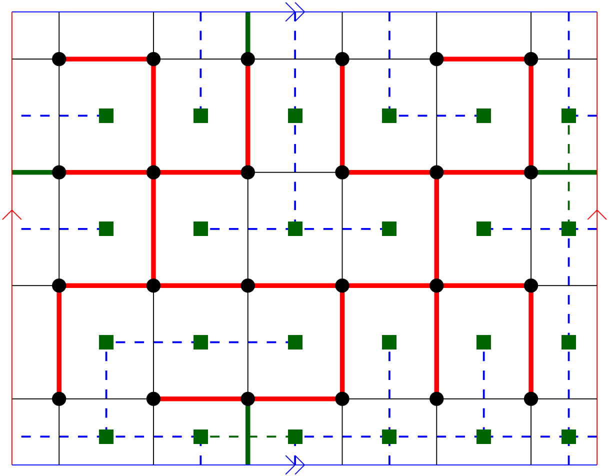
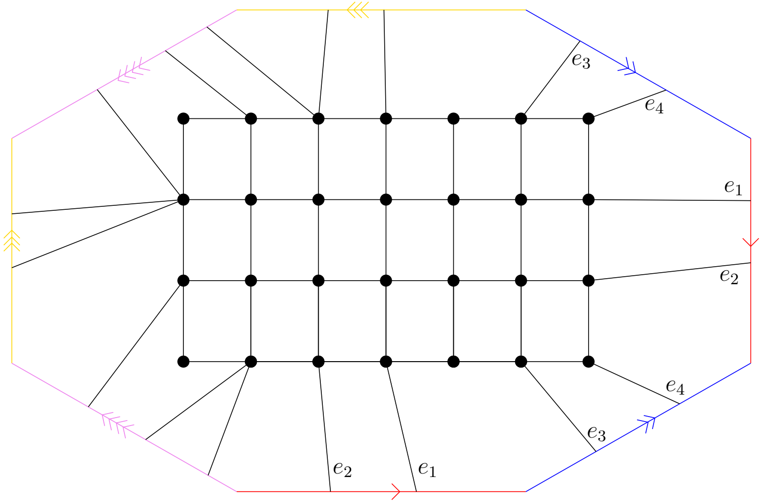


Tree-cotree partition - Example 2



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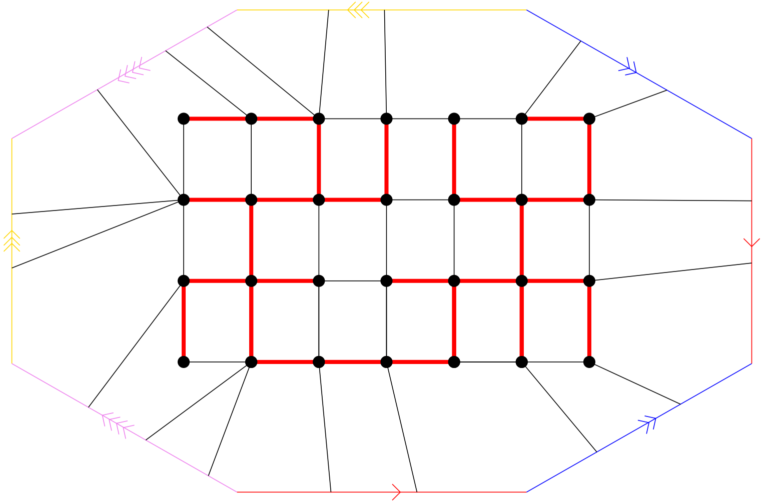

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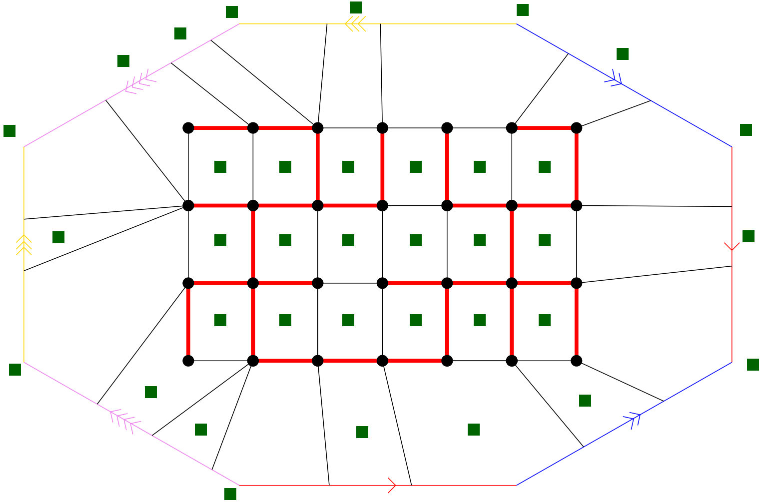

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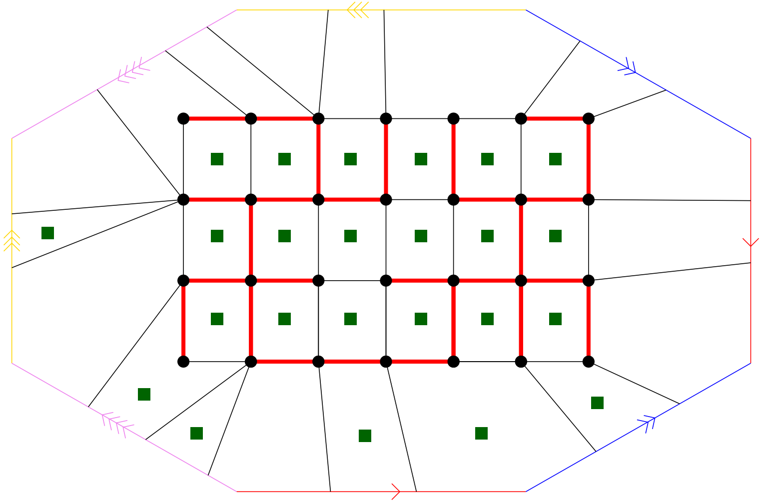

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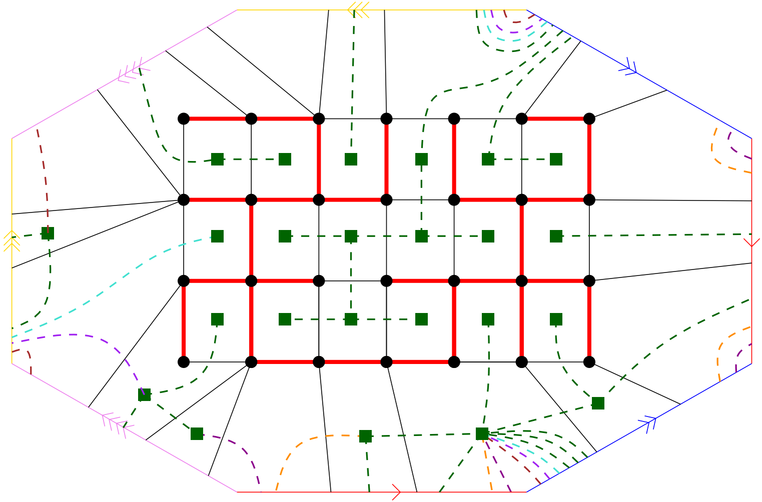

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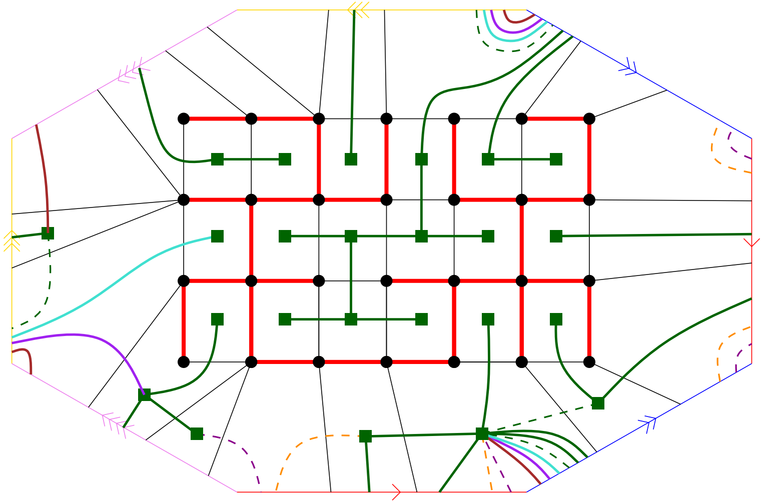

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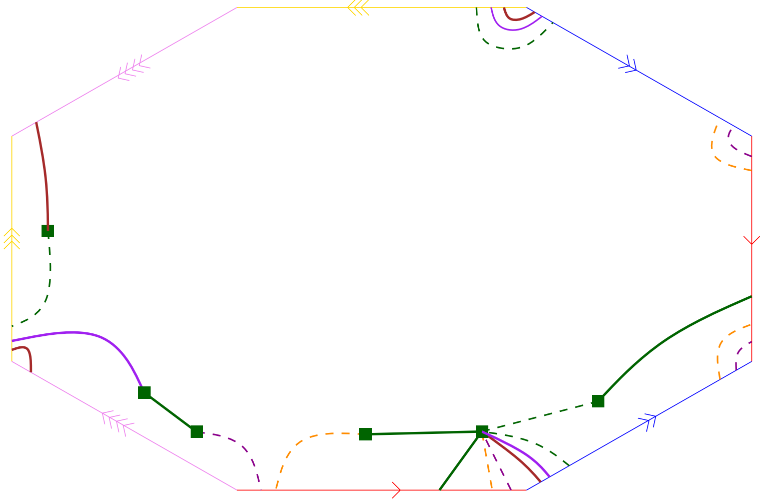

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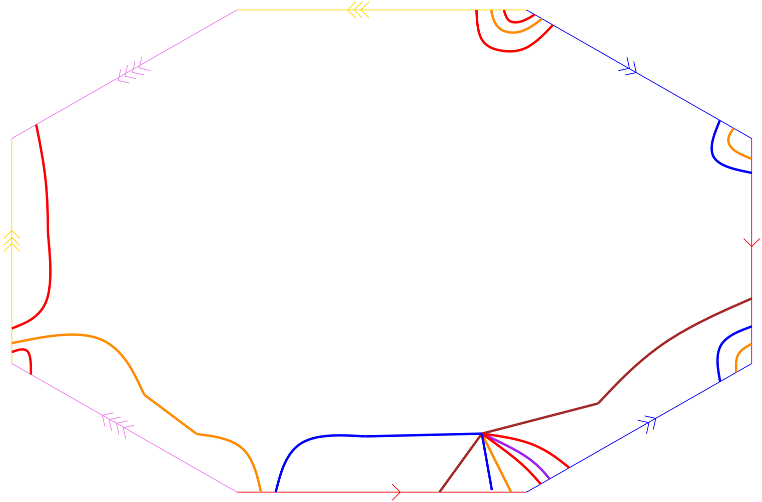

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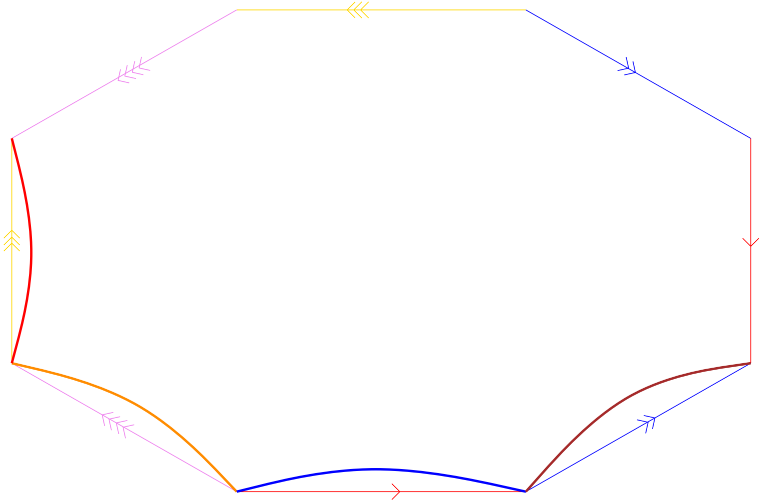

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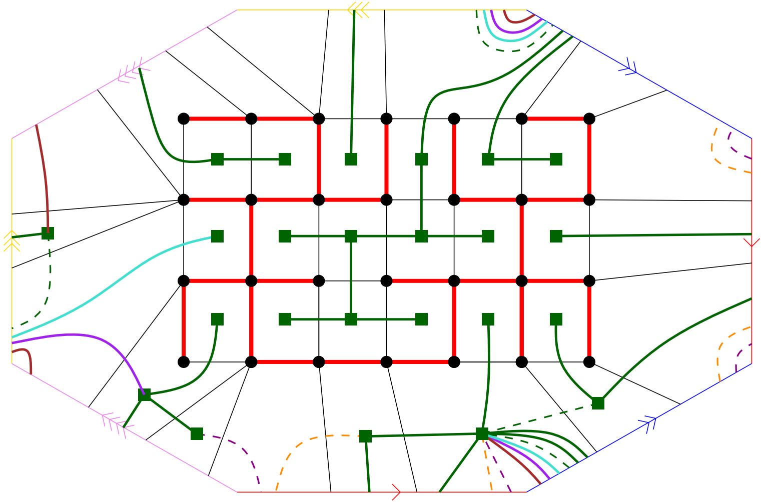

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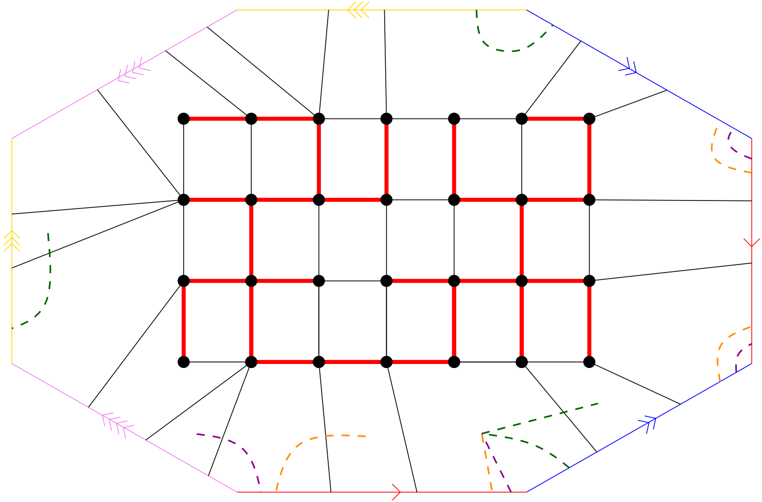

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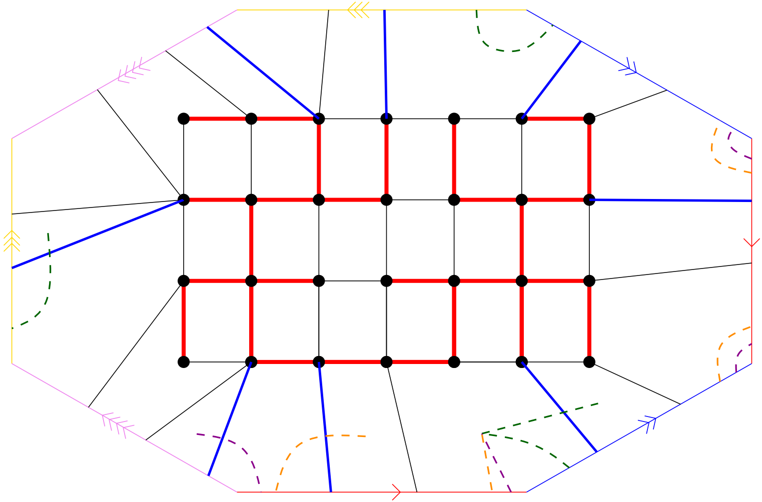

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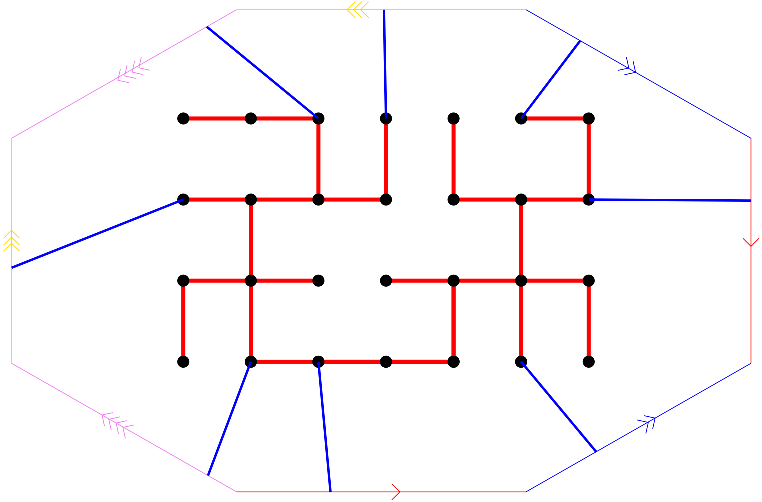

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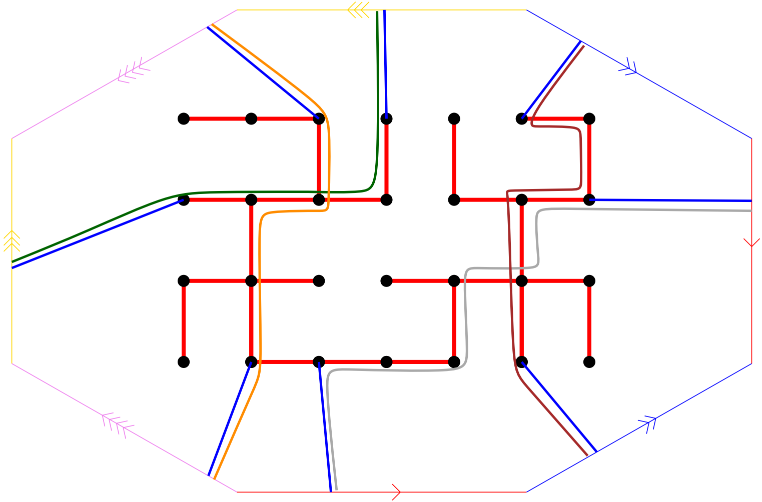

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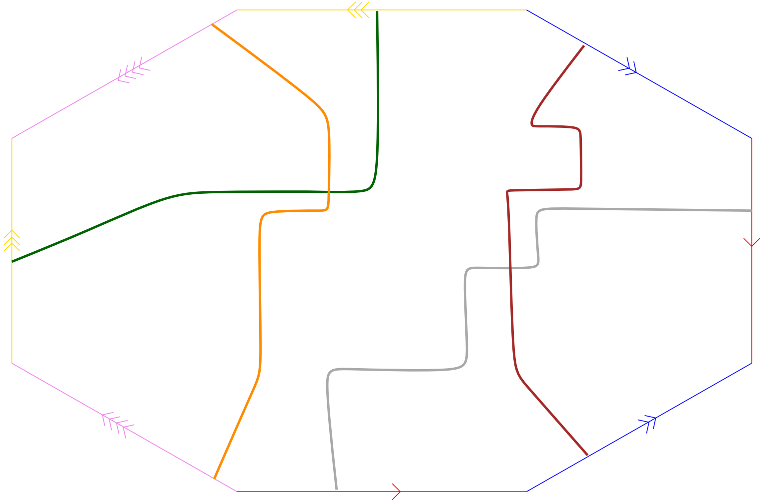
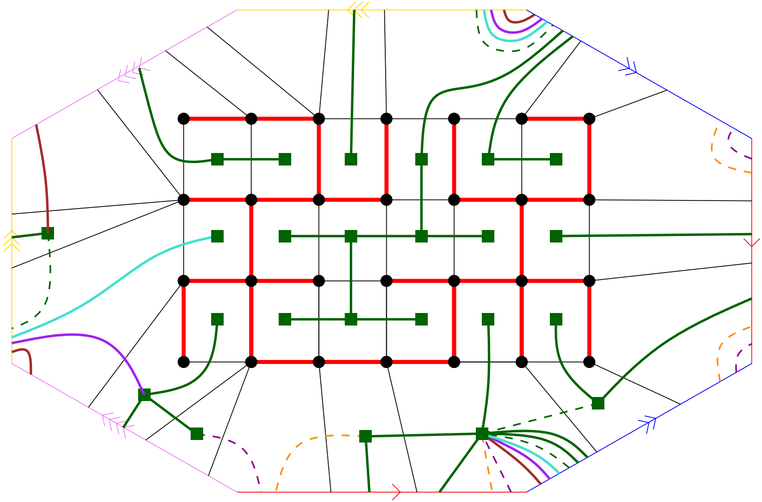


Tree-cotree partition - Cut graph

G embedded graph
$H \subset G$ a cut graph if G✄H is planar

- (T, C, X) is a tree-cotree partition of G
- $T \cup X$ is a cut graph: join faces according to C^*
- By duality, $C^* \cup X^*$ is a cut graph

Cut graph - Example



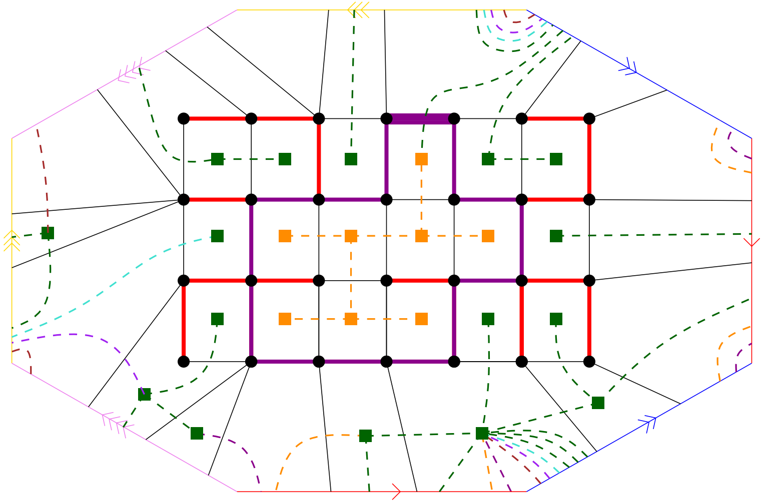
Tree-cotree partition - Nice loops

G embedded graph
(T, C, X) is a tree-cotree partition of G
$A = C \cup X$
$e \in A$

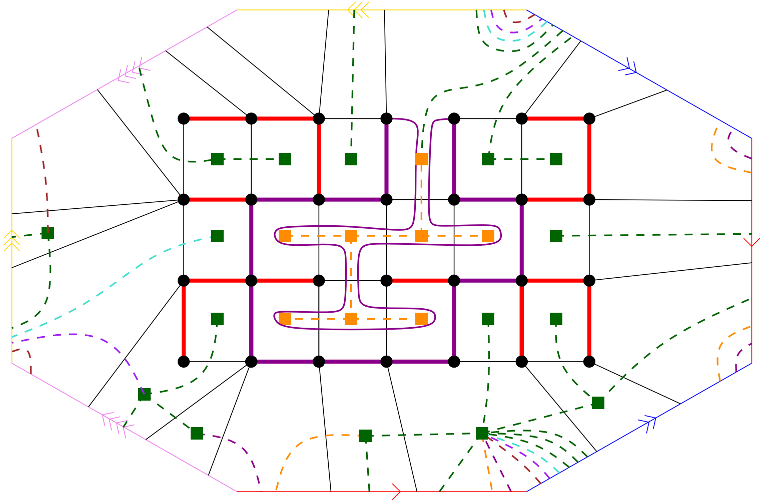
$\Rightarrow$ loop(T, e) contractible ifff $A^* - e^*$ has a tree component

- if loop(T, e) contractible $\Rightarrow$ loop(T, e) bounds a disk D $\Rightarrow$ $A - e$ contains a cotree of $G \cap D$
- if $A - e$ contains a cotree of $G \cap D$ $\Rightarrow$ deform e along $A^* - e^*$ $\Rightarrow$ cycle homotopic to $A - e$ loop(T, e) disjoint from A^* $\Rightarrow$ loop(T, e) contractible

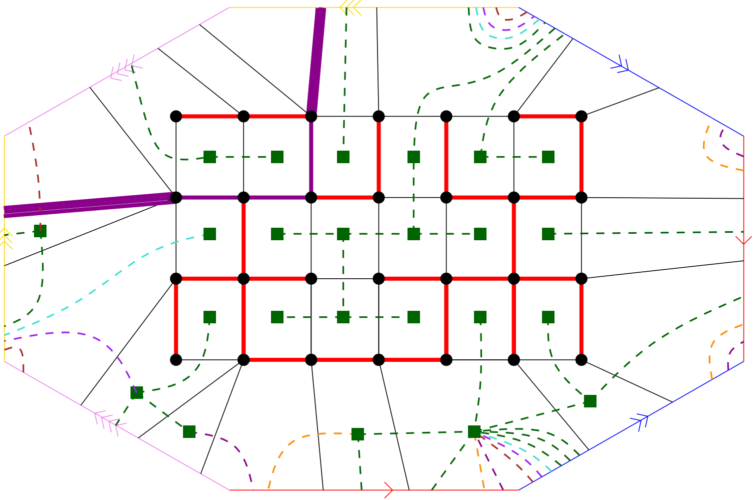
Nice loops - Contractible



Nice loops - Contractible



Nice loops - Contractible

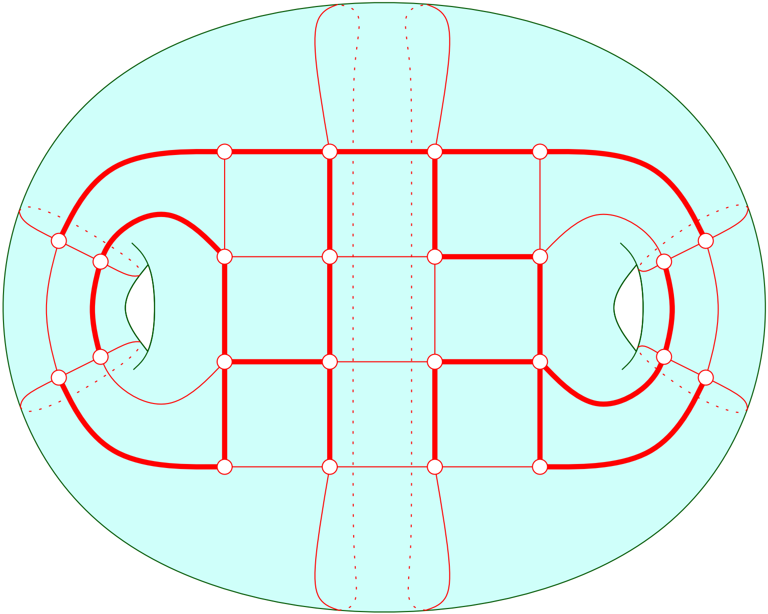


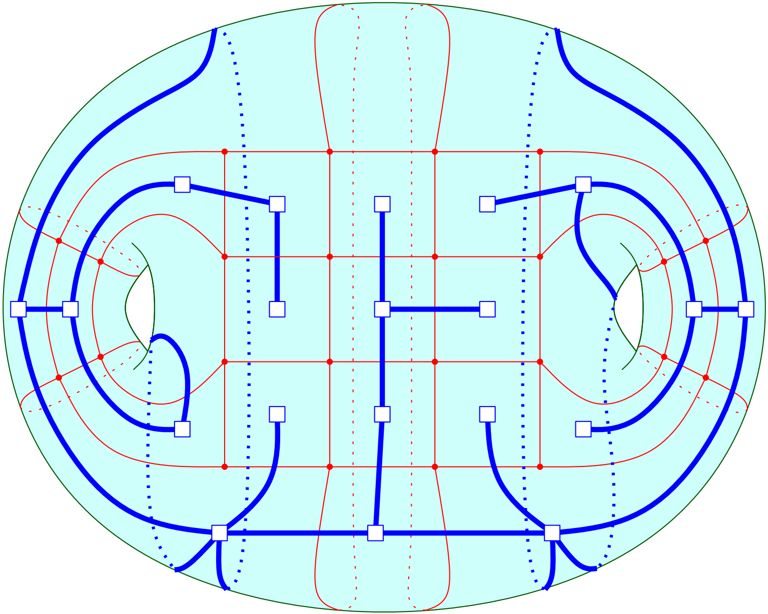
Tree-cotree partition - Nice loops

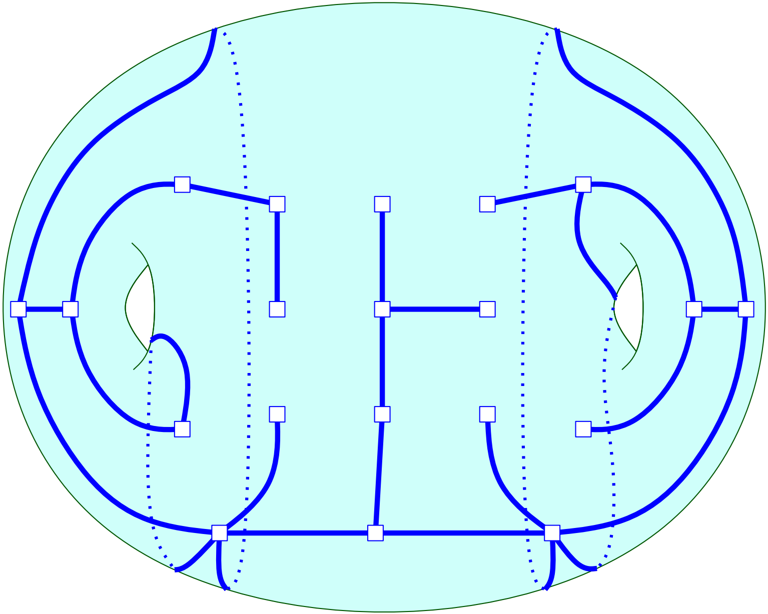
G embedded graph
(T, C, X) is a tree-cotree partition of G
$A = C \cup X$
$e \in A$

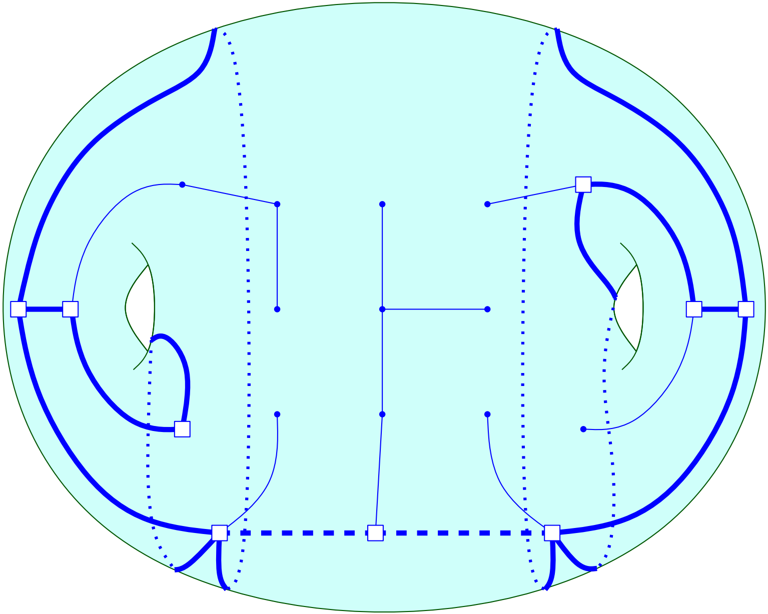
$\Rightarrow$ loop(T, e) separating ifff $A^* - e^*$ disconnected

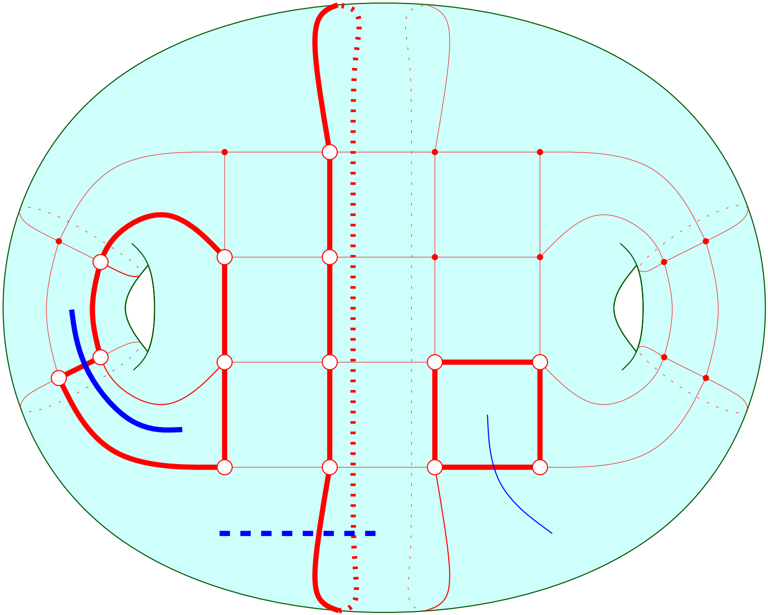
- $A^* - e^*$ gives a way to merge faces











Shortest non-contractible loop

G embedded graph
$x \in V(G)$
L_x contractible loops from x Compute shortest loop outside L_x

- compute shortest path tree T from x
- compute dual $A^* = G^* - E(T)^*$
- compute $B = \{e \in A \mid A^* - e^* \text{ has no tree-component}\}$
- compute

$$e = \arg\min_{uv \in B} \{d_T(x,u) + d_T(x,v) + |uv|\}$$

- return loop(T, e)

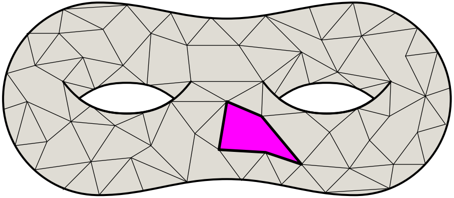
$\Rightarrow$ linear time per vertex x

Representation of some distances

Theorem

Let f be a specified face in an embedded graph G.
Preprocess G in $O(g^2 n \log n)$ time such that:

$$\textit{query } (u,v) \in V \times f \quad \xrightarrow{O(\log n)\ \textit{time}} \quad \textit{distance } d_G(u,v)$$



- compute sp-tree (shortest path) at one vertex
- iteratively move to the neighbor in the face and update the sp-tree

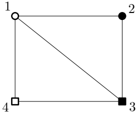
Representation of some distances - Planar

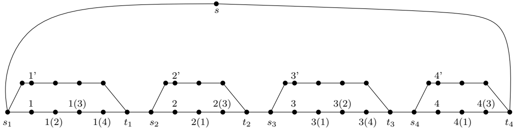
Approach for planar graphs

- compute sp-tree at one vertex of the face
- iteratively move to the neighbor in the face and update the sp-tree
- efficient dynamic data structures to detect what edges come in and out
- reminiscence of *kinetic* data structures
- use of tree-cotree decomposition
- each (directed) edge appears in a contiguous family of sp-trees (via crossing argument)
- persistence

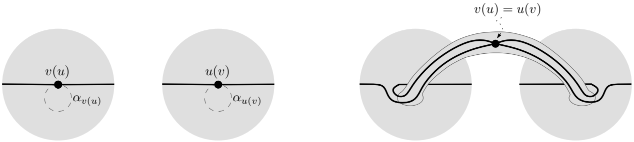
Shortest separating cycle

- max independent set reduces to:
  shortest cycle in planar graph with forbidden pairs





- surgery to represent the forbidden pairs



- separating cycle $\Leftrightarrow$ crosses any closed curve even nb of times

Outline

- Topology and graphs on surfaces
- Algorithmic problems in embedded graphs
- Sample of techniques
- FPTness of crossing number
- Stretch

FPTness crossing number

- Input: graph G and integer $k > 0$
- Parameter: k
- Question: $cr(G) \leq k$?
  - Solvable in $O(f(k) \cdot n^2)$ time [Grohe '04]
  - Solvable in $O(f(k) \cdot n)$ time [Kawarabayashi and Reed '07]
  - for each constant k, linear time

FPTness crossing number – Ingredients

1. Embedding in surface of genus $g = k$
2. face-width $\geq a(k) \Rightarrow$ crossing number $> k$
3. find a subset $A \subset V(G)$ of $b(k) = k \cdot a(k)$ such that $H = G - A$ planar
4. while treewidth of $H = G - A$ is $\geq t(k) = 4000k^2$
   - H has a $(600k^2)$-grid minor
   - inside there is a flat $(6k)$-grid minor of G
   - inside a flat $(6k)$-grid minor the middle $(2k)$-grid minor is irrelevant
   - find and remove irrelevant vertices
5. when treewidth of $H \leq t(k) \Rightarrow$ use MSO on $H + A$
   - $H + A$ has treewidth $t(k) + b(k)$

Small treewidth

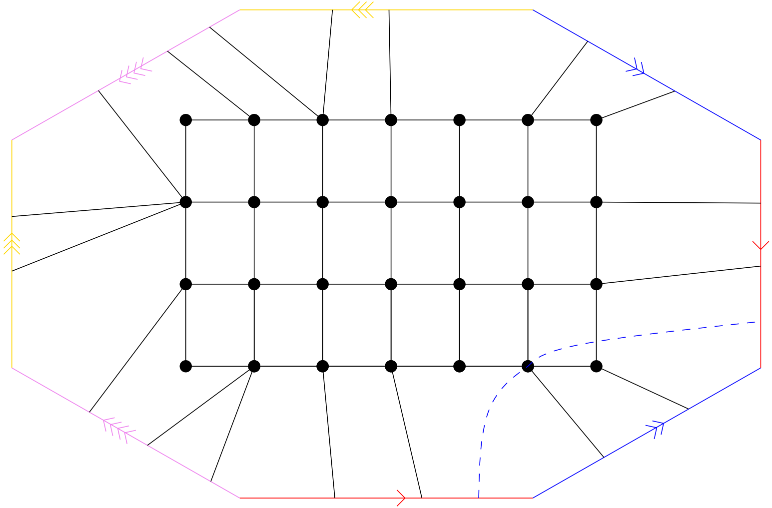
Input: integer $k > 0$ and a graph H with n vertices and treewidth $f(k)$.
Paremeter k
Question: is $cr(H) \leq k$?

- Solvable in $O(n)$ time for each fixed k
- Monadic second order expression
- Courcelle's theorem MSO

Facewidth

- G embedded in Σ
- facewidth $fw(G)$ is $\min cr(\gamma, G)$ over all non-contractible curves γ on Σ
- $\sim$ facial distance
- measure of local planarity
- $\frac{1}{2}$ shortest non-contractible cycle in vertex-face incidence graph

Example facewidth 1



Large facewidth $\Rightarrow$ Large crossing number

Theorem

There is some $a(k)$ *such that* $fw(G) \geq a(k) \Rightarrow cr(G) \geq k$

- $fw(G) \geq a(k)$ implies G has a $C_k \square C_k$ minor
  [minors][Brunet, Mohar, Richter '96]
- $cr(C_k \square C_k) \geq k$
- H a minor of G, $\Delta(H) \geq 4 \Rightarrow cr(G) \geq cr(H)/4$
  [Garcia-Moreno, Salazar '01]

Constructing A – Cutting

1. $H := G$
2. repeat until H is planar
   2.1 If $fw(H) \geq a(k)$, reply "$cr(G) > k$"
   2.2 Else
      2.2.1 Take a curve γ defining $fw(H)$
      2.2.2 Remove in H vertices in $\gamma \cap V(H)$
      2.2.3 Cut Σ along γ and attach disks to the boundaries

- we end up with H planar
- the set A of removed vertices has $\leq g \cdot a(k) = b(k)$ vertices

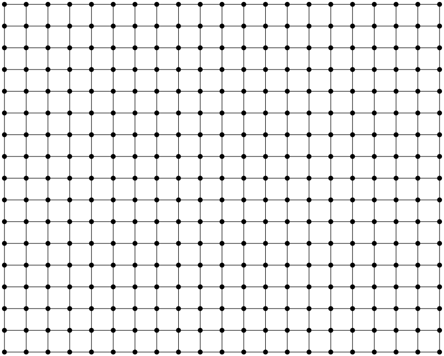
FPTness crossing number – Ingredients

1. *Done!* Embedding in surface of genus $g = k$
2. *Done!* face-width $\geq a(k) \Rightarrow$ crossing number $> k$
3. *Done!* find a subset $A \subset V(G)$ of $b(k) = k \cdot a(k)$ such that $H = G - A$ planar
4. while treewidth of $H = G - A$ is $\geq t(k) = 4000k^2$
   - H has a $(600k^2)$-grid minor
   - inside there is a flat $(6k)$-grid minor of G
   - inside a flat $(6k)$-grid minor the middle $(2k)$-grid minor is irrelevant
   - find and remove irrelevant vertices
5. *Done!* when treewidth of $H \leq t(k) \Rightarrow$ use MSO on $H + A$
   - $H + A$ has treewidth $t(k) + b(k)$

Grid minor

Theorem (Robertson, Seymour, Thomas '94)

If H is planar and $tw(G) \geq 4000k^2 \Rightarrow H$ has a $(600k^2)$-grid minor and can be found in $O(f(k) \cdot n)$ time

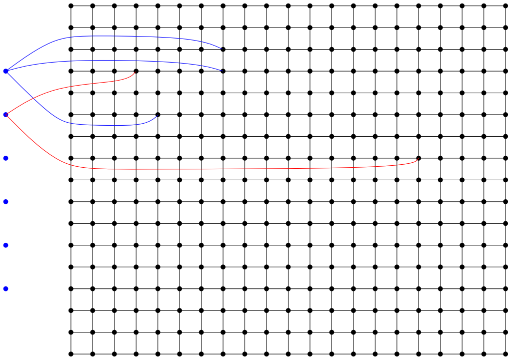


Grid minor

Theorem (Robertson, Seymour, Thomas '94)

If H is planar and $tw(G) \geq 4000k^2 \Rightarrow H$ has a $(600k^2)$-grid minor and can be found in $O(f(k) \cdot n)$ time

- when adding A there are non-planar parts

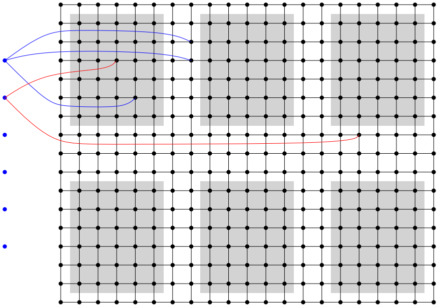


Flat grid minor

Theorem (Thomassen '97)

If G has max genus k and a $(600k^2)$-grid minor J, then there is a flat $(6k)$-grid minor $J' \subset J$. It can be found in $O(f(k) \cdot n)$ time.

- consider $2k+2$ disjoint subgrids of J
- if none of them flat, then $genus(G) > k$

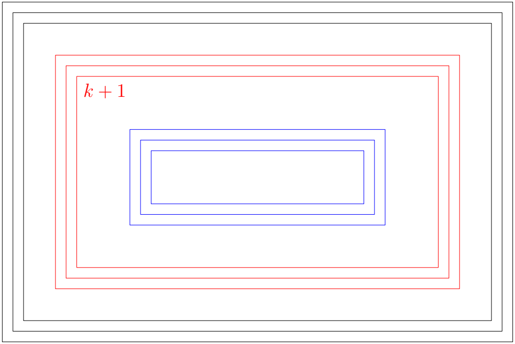


Irrelevant vertices

Lemma

If G has a flat $(6k)$-grid minor J' and $cr(G) \leq k$ then the middle $(2k)$-grid and its attachments do not participate in any crossing.

- one of the middle $k+1$ grid cycles has no crossings

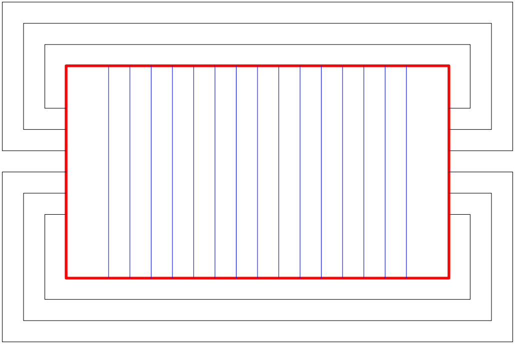


Irrelevant vertices

Lemma

If G has a flat $(6k)$-grid minor J' and $cr(G) \leq k$ then the middle $(2k)$-grid and its attachments do not participate in any crossing.

- the exterior of that cycle cannot be drawn inside without producing k^2 crossings



Irrelevant vertices

Lemma

If G has a flat $(6k)$-grid minor J' and $cr(G) \leq k$ then the middle $(2k)$-grid and its attachments do not participate in any crossing.

- in any drawing of $G - J'$ we can redraw J' without crossings

Irrelevant vertices

Lemma

If G has a flat $(6k)$-grid minor J' and $cr(G) \leq k$ then the middle $(2k)$-grid and its attachments do not participate in any crossing.

- finding and removing irrelevant regions doable in $O(k^2)$ amortized time

FPTness crossing number – Ingredients

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   - H has a $(600k^2)$-grid minor
   - inside there is a flat $(6k)$-grid minor of G
   - inside a flat $(6k)$-grid minor the middle $(2k)$-grid minor is irrelevant
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Crossings of cycles

- G a graph embedded in orietable Σ
- α and β cycles in G
- $cr(\alpha, \beta) = \min cr(\alpha', \beta)$ over all tiny deformations α' of α
- $cr_2(\alpha, \beta) = cr(\alpha, \beta) \mod 2$
- computing $cr(\alpha, \beta)$ is not obvious
- computing $cr_2(\alpha, \beta)$ is easy
  - invariant under tiny deformations

Stretch – Definition

- G a graph embedded in orientable Σ
- stretch is

$$\min |\alpha| \cdot |\beta|$$

  over all cycles α and β with $cr(\alpha, \beta) = 1$
- original definition via 1-leaping
- introduced by Chimani and Hliněný
  related to lower bound for crossing number of G embedded in surface
- here: computing it in $O(16^g g^2 n \log n)$ time
  Cabello, Chimani, Hliněný, Štefankovič TBW-TBS-TBP

Stretch – Modulo 2

- G a graph embedded in orientable Σ
- stretch is
$$\text{min}\, |\alpha| \cdot |\beta|$$
over all cycles α and β with $cr(\alpha, \beta) = 1$
- stretch is also
$$\text{stretch}_2 = \text{min}\, |\alpha| \cdot |\beta|$$
over all cycles α and β with $cr_2(\alpha, \beta) = 1$
- let (α^*, β^*) be the pair attaining stretch$_2$
  - if they cross ≥ 2, uncrossing argument gives a better stretch$_2$

1-cycles

Introduction to $\mathbb{Z}_2$ homology

- a 1-cycle γ is a subset of edges with even degree
- an even subgraph
- union of
- symmetric sum $\oplus$ is nice operation between 1-cycles

$$\gamma \oplus \alpha = \{e \in E(\gamma) \cup E(\alpha) \mid e \notin \gamma \text{ or } e \notin \alpha\}$$

- set of 1-cycles Z_1 is a vector space over $\mathbb{Z}_2$
- each 1-cycle is the union of some (graph-theory) cycles
- $cr_2(\, , \,)$ meaningful for 1-cycles
  independent of decomposition into cycles
  not possible for $cr(\, , \,)$

Stretch – Modulo 2 and 1-cycles

- $\mathrm{stretch}_2$ is also

$$\mathrm{stretch}_{2,1-cycle} = \min |\alpha| \cdot |\beta|$$

  over all 1-cycles α and β with $cr_2(\alpha, \beta) = 1$
- let (α^*, β^*) be the pair attaining $\mathrm{stretch}_{2,1-cycle}$
- $\alpha^* = \gamma_1 \oplus \cdots \oplus \gamma_k$ where each γ_i cycle
- $\beta^* = \sigma_1 \oplus \cdots \oplus \sigma_t$ where each σ_j cycle

$$1 = cr_2(\alpha^*, \beta^*) = \sum_{i,j} cr_2(\gamma_i, \sigma_j)$$

- $cr_2(\gamma_i, \sigma_j) = 1$ for some i and j
  γ_i, σ_j define smaller $\mathrm{stretch}_{2,1-cycle}$

Boundary 1-cycles – Homology

- A 1-cycle α is a boundary cycle if $\alpha = f_1 \oplus \cdots \oplus f_k$ for some facial walks $f_1, \ldots, f_k$.
- set of 1- boundaries B_1 form a vector space over $\mathbb{Z}_2$
- $B_1 \subseteq Z_1$
- $H_1 := Z_1/B_1$ homology group
- $[0] = B_1$
- for 1-cycle α, the class $[\alpha]$ is

$$\{\beta \in Z_1 \mid \beta = \alpha \oplus f_1 \oplus \cdots \oplus f_k \text{ for some } f_1, \ldots, f_k\}$$

$$\{\beta \in Z_1 \mid \beta = \alpha \oplus \gamma, \gamma \in B_1\}$$

Crossings of 1-cycles – Homology

- $cr_2(\alpha \oplus \alpha', \beta) = cr_2(\alpha, \beta) + cr_2(\alpha', \beta)$
- $\alpha \in B_1 \Rightarrow cr_2(\alpha, \beta) = 0$
- $cr_2(\alpha, \beta')$ is invariant over all $\beta \in [\beta]$
- $cr_2([\alpha], [\beta]) := cr_2(\alpha, \beta)$ is well defined
- not so nice properties for $cr(\ )$

Crossings of 1-cycles – Properties

1. *stretch* $= \infty$
2. for each homology classes $[\alpha]$ and $[\beta]$
   2.1 find shortest 1-cycle $\alpha' \in [\alpha]$
   2.2 find shortest 1-cycle $\beta' \in [\beta]$
   2.3 if α' and β' cycles,
   $cr_2(\alpha, \beta) = 1$ AND
   $|\alpha'| \cdot |\beta'| < stretch$
   THEN $stretch := |\alpha'| \cdot |\beta'|$

Shortest 1-cycles in each homology class computable in $O(16^g g^2 n \log n)$

[Erickson,Nayyeri '11]

(There are 2^g homology classes.)

Conclusions

- Topology and graphs on surfaces
- Algorithmic problems in embedded graphs
- Sample of techniques
- FPTness of crossing number
- Stretch