Crossings of near-planar graphs

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(based on joint work with Bojan Mohar)

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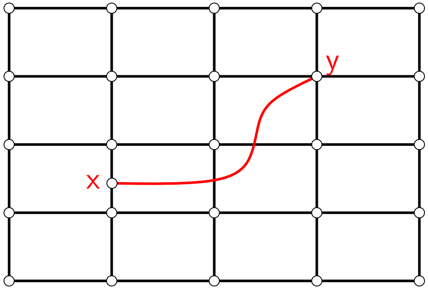
Outline

Near-planar graphs

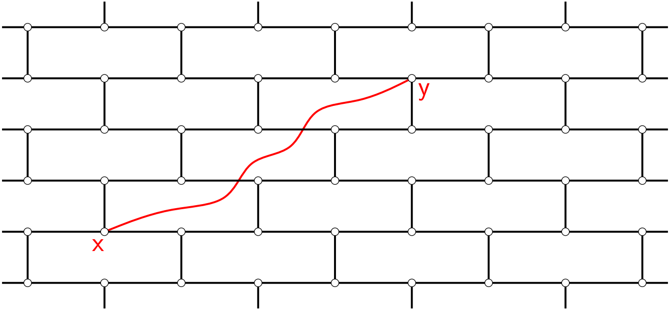
Non-planar H is near-planar if $H = G + xy$ for planar G



- weak relaxation of planarity
- near-planar $\subsetneq$ toroidal, apex

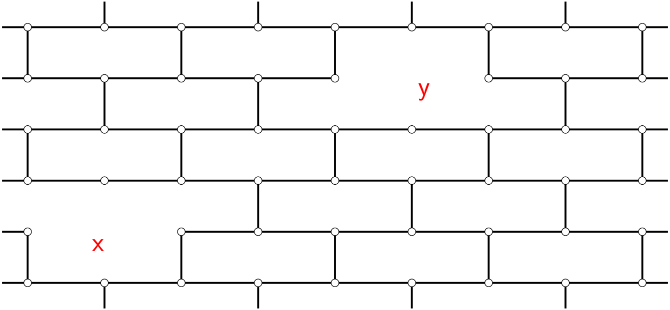
Near-planar – Riskin

- G planar, 3-connected, and 3-regular [Riskin '96]
  - $cr(G + xy)$ attained by the following drawing: draw G planarly (unique) and insert xy minimizing crossings



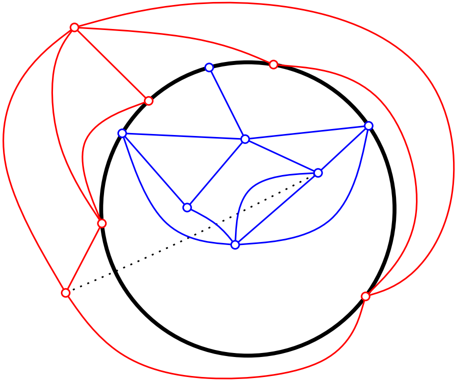
Near-planar – Riskin

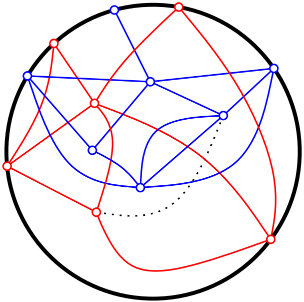
- G planar, 3-connected, and 3-regular [Riskin '96]
  - $cr(G + xy)$ attained by the following drawing: draw G planarly (unique) and insert xy minimizing crossings
  - $cr(G + xy)$ is a distance in $(G - x - y)^*$



Near-planar – Riskin

- No extension to non-cubic graphs possible [Mohar '06]
  also [Farr],[Hliněný, Salazar '06]





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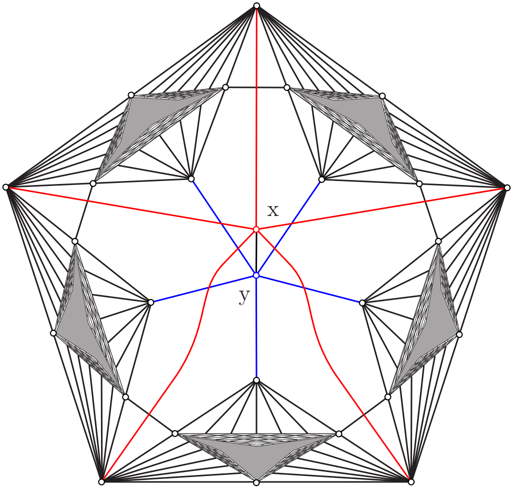


Near-planar – Objective

understanding near-planar graphs

Near-planar – Objective

understanding near-planar graphs

- combinatorial properties
- crossing number
- 1-planarity

Outline

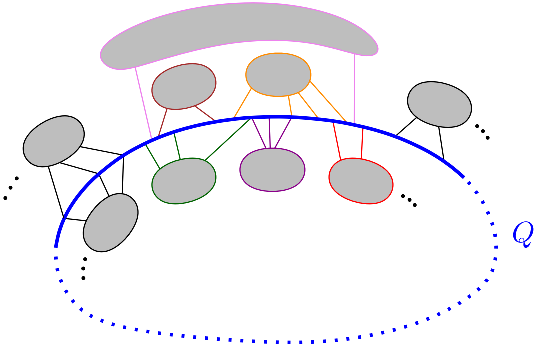
1. Near-planar graphs
2. Planar separability
3. Dual and facial distances
4. Approximating crossing number
5. Hardness crossing number
6. 1-planarity

Planar separability

- G a planar graph
- $x, y \in V(G)$ distinct
- $Q \subset G - x - y$
- Q planarly separates x and y if
  - in each embedding Γ of G each (x, y)-arc intersects Q
  - in each embedding Γ of G, x and y in different faces of $\Gamma(Q)$

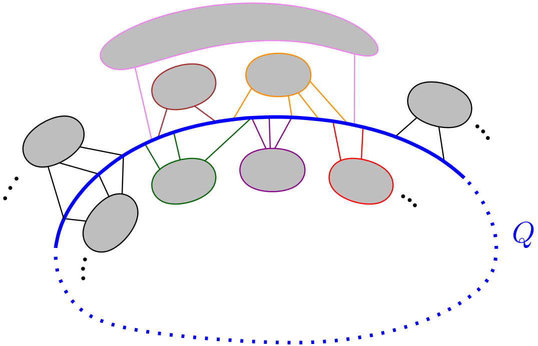
Planar separability – Bridges

- G a planar graph
- $Q \subset G$ a cycle
- Q-bridges
  - edges $\notin Q$ joining vertices in Q
  - connected components of $G - Q$ with edges of attachment



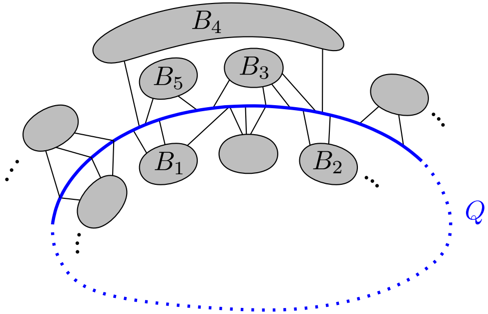
Planar separability – Overlapping bridges

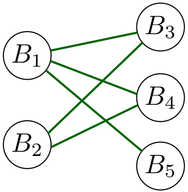
- G a planar graph
- $Q \subset G$ a cycle
- two bridges overlap iff
  - they have 3 common vertices of attachment
  - each has 2 vertices of attachment alternating along Q



Planar separability – Overlap graph

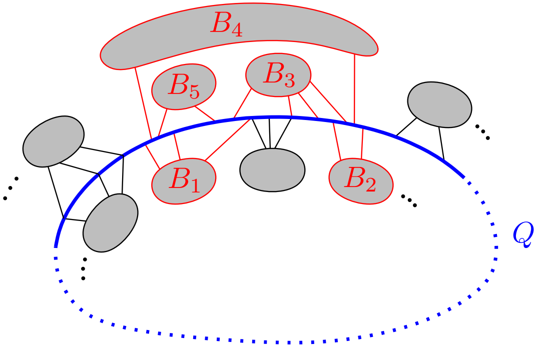
- G a planar graph
- $Q \subset G$ a cycle
- *overlap*(G, C)
  - vertices are Q-bridges
  - edges between overlapping Q-bridges





Planar separability – Separating cycle

- G a planar graph. $x, y \in V(G)$
- $Q \subset G - x - y$ a cycle
- $B_x(Q)$ is Q-bridge containing x
- Q planarly separates x and y $\Leftrightarrow$ $B_x(Q)$ and $B_y(Q)$ weakly overlap



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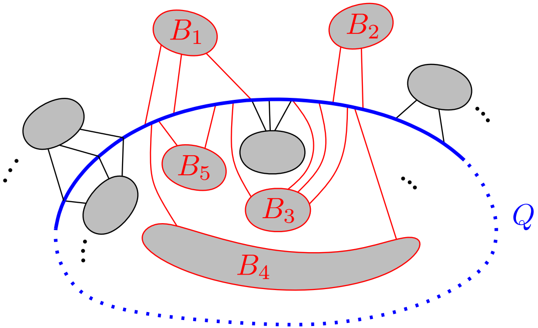
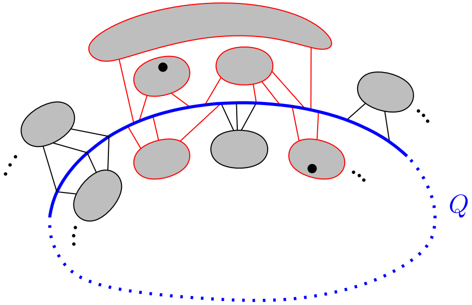


Planar separability – Tutte

- G a planar graph. $x, y \in V(G)$
- $Q \subset G - x - y$ a cycle that planarly separates x and y
- $B_x(Q)$ and $B_y(Q)$ weakly overlap
- exits cycle $Q' \subset G - x - y$ such that $B_x(Q')$ and $B_y(Q')$ overlap
- Q' edge-disjoint from $B_x(Q)$ and $B_y(Q)$



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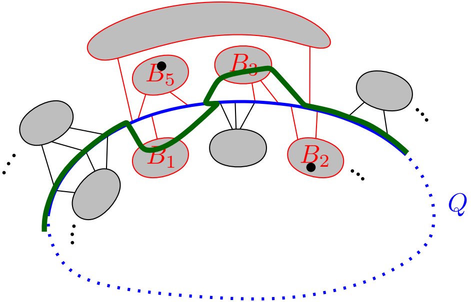

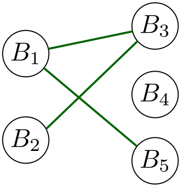

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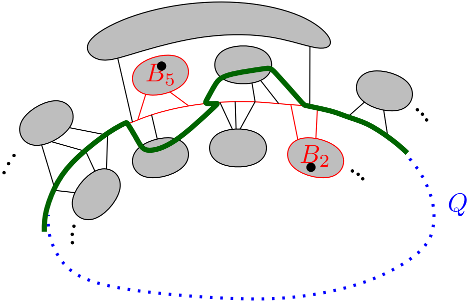

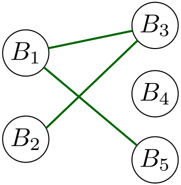

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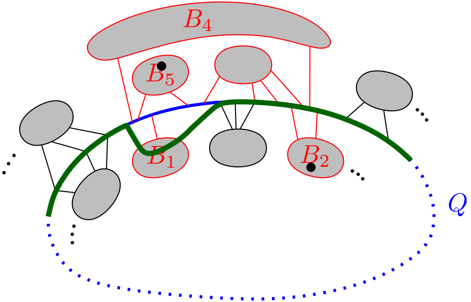

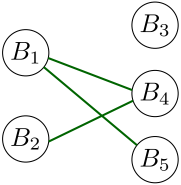

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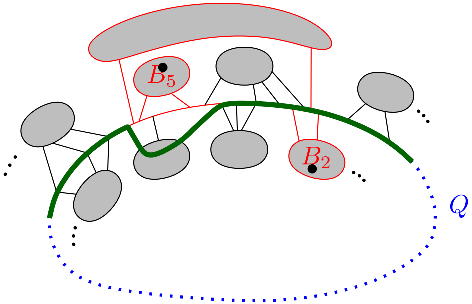
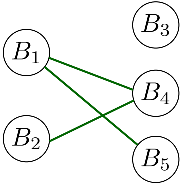




Planar separability – Tutte

Theorem

[Tutte '75]

- *G a planar graph*
- *$x, y \in V(G)$ distinct*

$G + xy$ non-planar $\Leftrightarrow$ exists cycle $Q \subset G - x - y$ s.t. $B_x(Q')$ and $B_y(Q')$ overlap

Planar separability – An extension

Theorem

- *G planar graph*
- *$x, y \in V(G)$ distinct*
- *$Q \subset G - x - y$ planarly separates x and y*

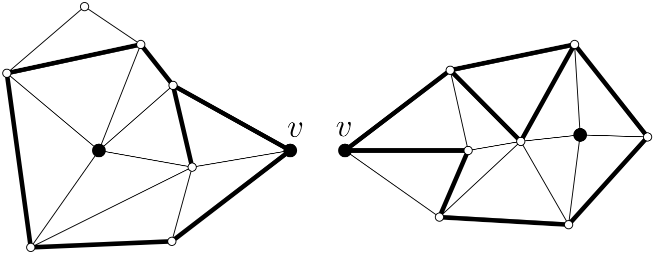
$\Rightarrow$ *exists cycle $Q' \subset Q$ that planarly separates x and y*

Planar separability – Connectivity reductions

$G + xy$ 2-connected $\Rightarrow$ G 2-connected

- $G = G_1 \cup G_2$ planar graph and $G_1 \cap G_2 = v$
- $x \in G_1$ and $y \in G_2$
- $Q \subset G - x - y$

Q planarly separates x and y $\Leftrightarrow$ $Q \cap G_1 - v$ separates x and v or $Q \cap G_2 - v$ separates x and v



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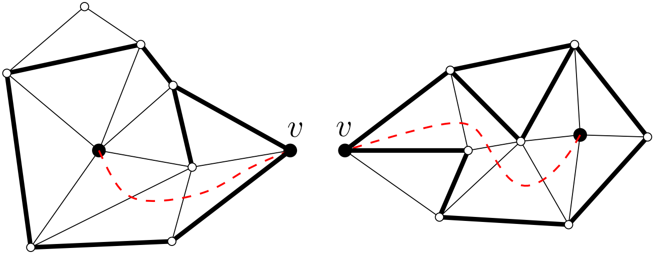

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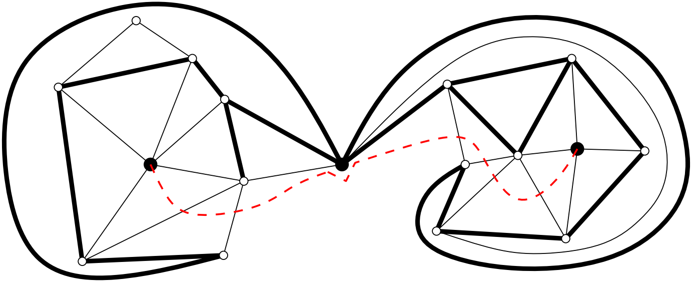
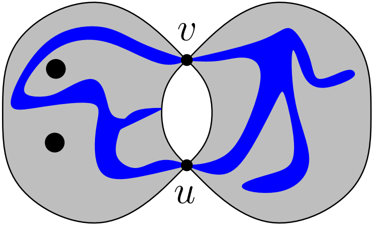


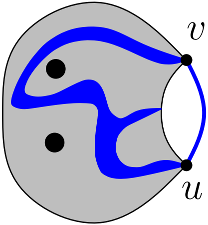
Planar separability – Connectivity reductions

G 2-connected $\Rightarrow$ $G + xy$ 3-connected

- $G = G_1 \cup G_2$ 2-connected planar graph and $G_1 \cap G_2 = \{u, v\}$
- $x, y \in G_1$
- $Q \subset G - x - y$
- $Q_1 = Q \cup G_1 + uv$ (if u-v connected in $G_2 \cap Q$) or
  $Q_1 = (Q \cup G_1) + uv$ (otherwise)

Q planarly separates x and y $\Leftrightarrow$ Q_1 separates x and y in $G_1 + uv$





Planar separability – Connectivity reductions

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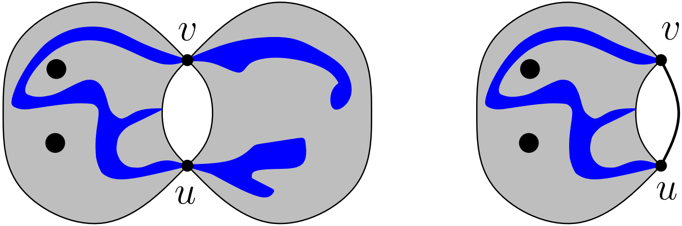
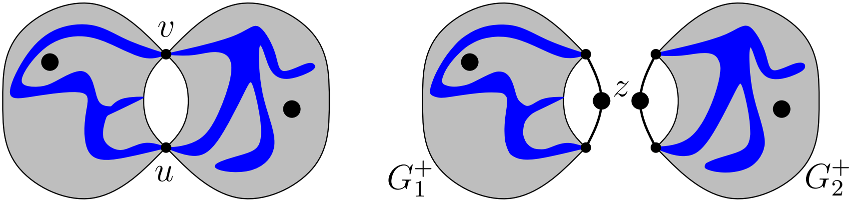


Planar separability – Connectivity reductions

$G + xy$ 3-connected $\Rightarrow$ G essentially 3-connected

- $G = G_1 \cup G_2$ 2-connected planar graph and $G_1 \cap G_2 = \{u, v\}$
- $x \in G_1$ and $y \in G_2$
- $Q \subset G - x - y$

Q planarly separates x and y $\Leftrightarrow$ $Q \cap G_1$ separates x and z in G_1^+ or $Q \cap G_2$ separates y and z in G_2^+



Outline

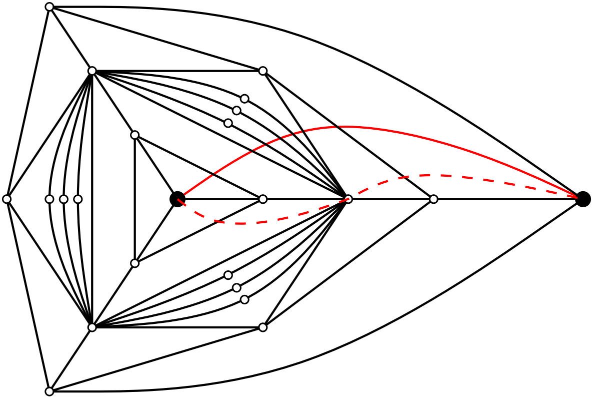
1. Near-planar graphs
2. Planar separability
3. Dual and facial distances
4. Approximating crossing number
5. Hardness crossing number
6. 1-planarity

Dual and facial distance – Plane

G plane graph (embedding in the plane is fixed)
x, y vertices of G.

- Dual distance of vertices x, y is
  $d^*(x, y) = \min\{cr(\gamma, G) \mid \gamma \text{ is an } (x, y)\text{-arc avoiding } V(G)\}$
- Facial distance between x and y is
  $d'(x, y) = \min\{cr(\gamma, G) \mid \gamma \text{ is an } (x, y)\text{-arc}\}$

Dual and facial distance – Plane



Dual and facial distance – Plane

G a plane graph (embedding in the plane is fixed)
x, y vertices of G.

- Dual distance $d^*(x,y)$ computable in linear time via dual graphs.
- Facial distance $d'(x,y)$ computable in linear time via face-vertex incidence graph.

Theorem (Riskin '96)

G planar, 3-connected, and 3-regular
$$\Rightarrow cr(G+xy) \quad = \quad d^*(x,y)$$

Dual and facial distance – Planar

G a planar graph (no embedding given)

- $d_0^*(x, y) = \min d^*(x, y)$ over all embeddings of G
  Computable in linear time [Gutwenger, Mutzel, Weiskircher '05]
  Alternative approach via connectivity reductions

- $d_0'(x, y) = \min d'(x, y)$ over all embeddings of G
  Computable in linear time via connectivity reductions

Dual and facial distance – Meaning

Theorem

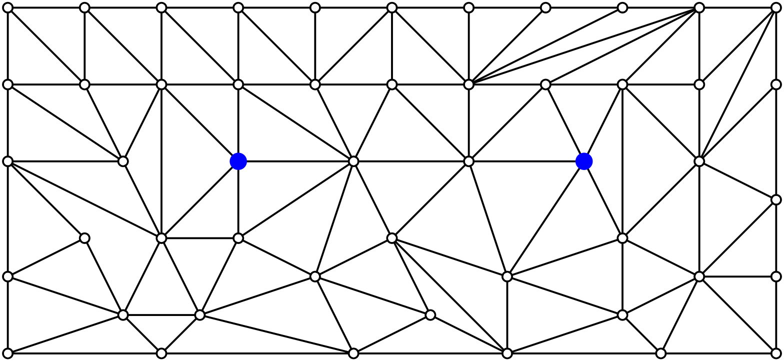
G planar graph

$x, y \in V(G)$

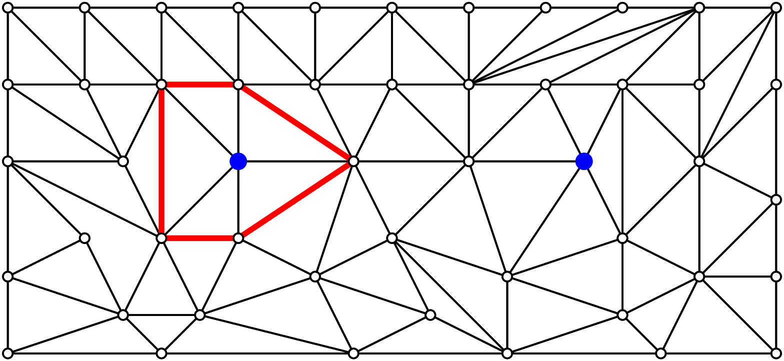
- $d_0^*(x,y)$ *is the maximum* r *such that:*
  G *has* r *edge-disjoint cycles planarly separating* x *and* y.
- $d_0'(x,y)$ *is the maximum* r *such that:*
  G *has* r *vertex-disjoint cycles planarly separating* x *and* y

This is easy if G essentially 3-connected because of unique embeddability.

Dual distance – 3-connected



Dual distance – 3-connected



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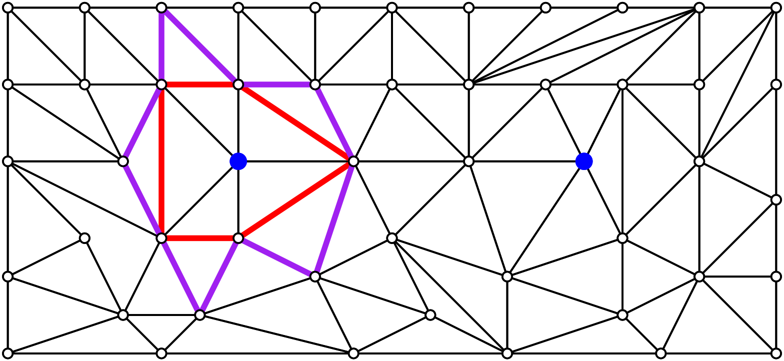

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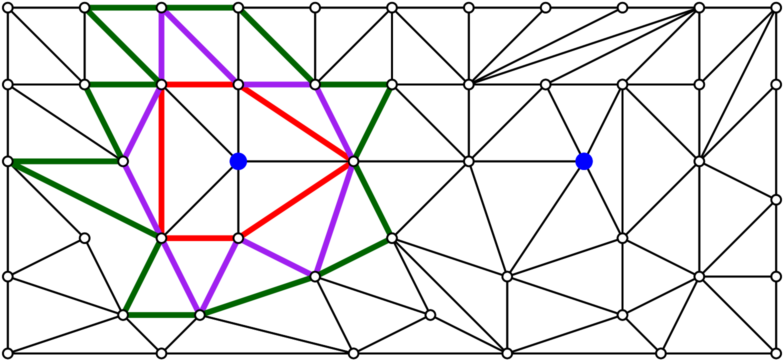

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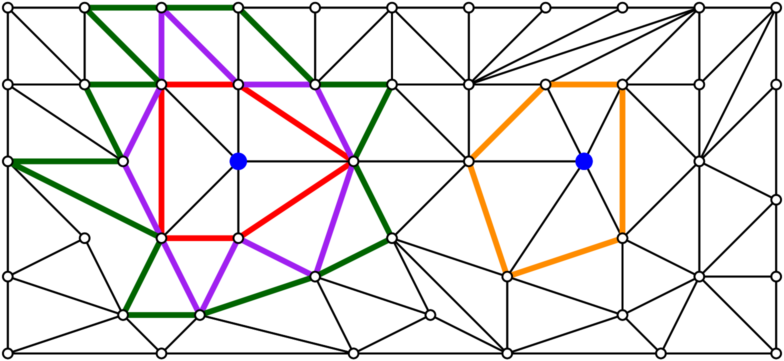

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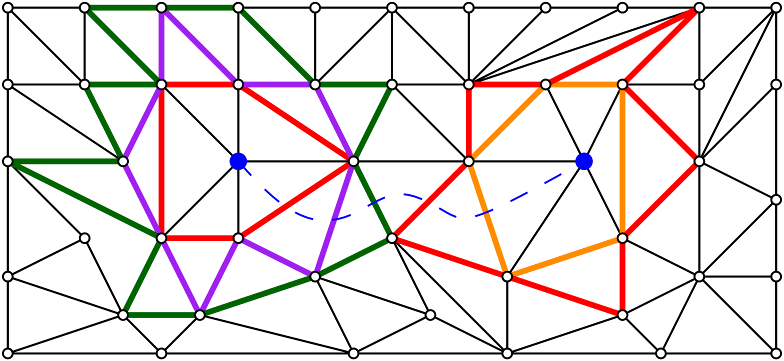
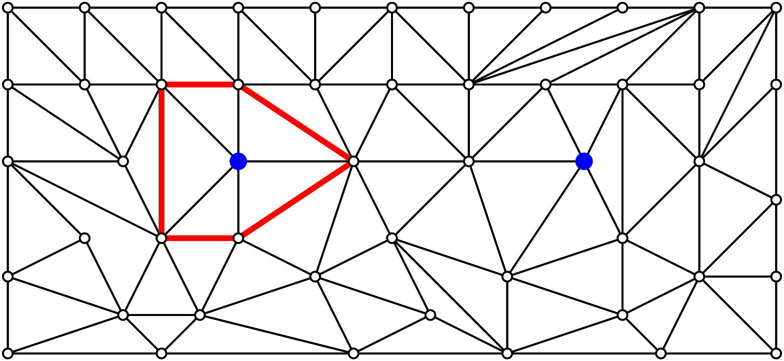
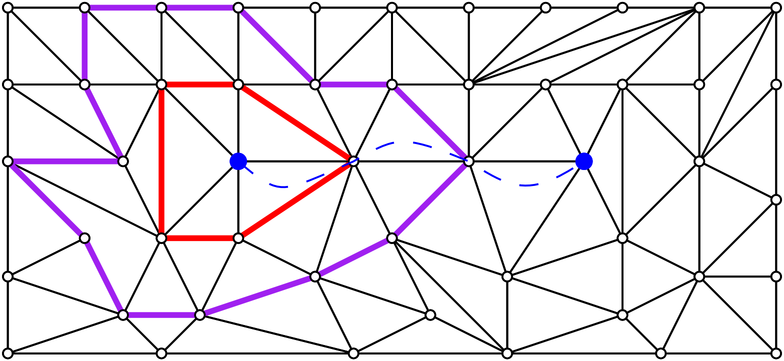


Facial distance – 3-connected

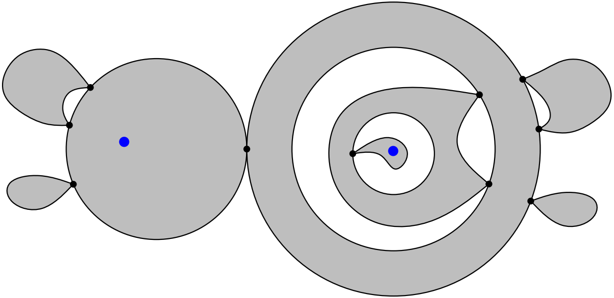


Facial distance – 3-connected



Dual and facial distance – General

For general G, use induction on $|V(G)|$ and connectivity reductions.



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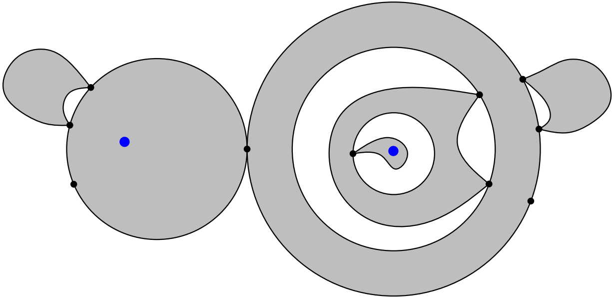

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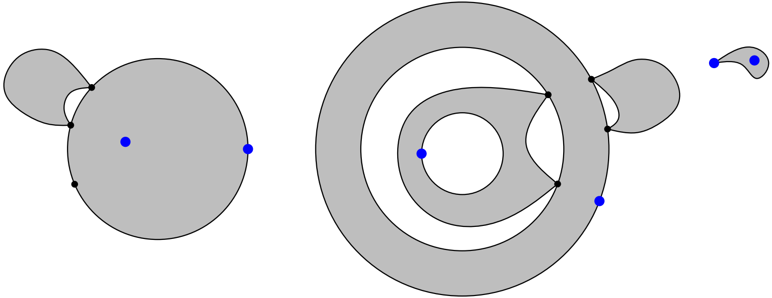

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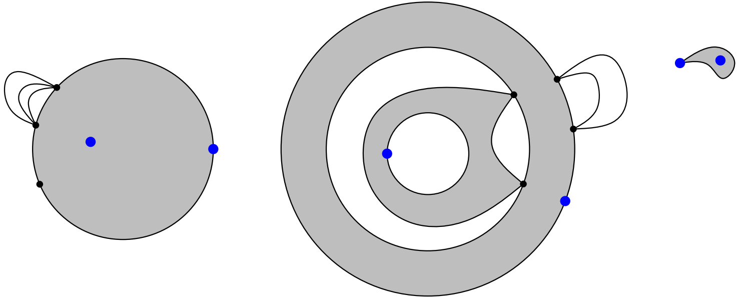

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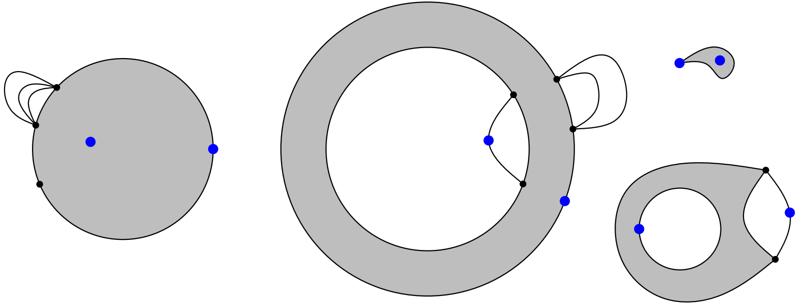
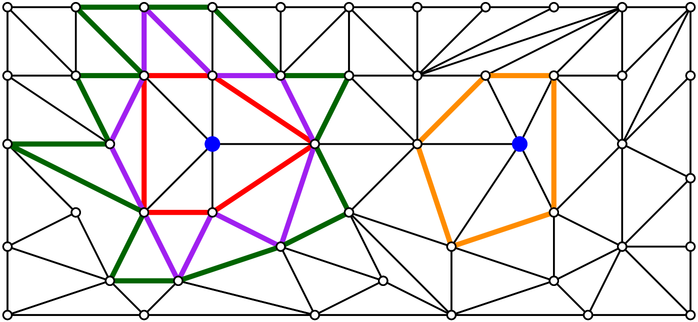


Nested cycles

- G plane graph
- Cycles Q_1 and Q_2 in G are nested in G if a small perturbation of Q_1 makes them disjoint.



Edge-disjoint nested cycles

- G plane graph
- x, y vertices of G
- $Q_1, Q_2 \subset G - x - y$ edge-disjoint cycles that planarly separate x and y

$\Rightarrow$ There are nested edge-disjoint cycles $Q_1', Q_2' \subset Q_1 \cup Q_2$ that planarly separate x and y

- if Q_1 and Q_2 nested, done
- connectivity reductions to assume essential 3-connectivity of G
- take any embedding Γ of G
- take Q_1' and Q_2' as the facial cycles of $\Gamma(Q_1 \cup Q_2)$ containing $\Gamma(x)$ and $\Gamma(y)$
- Q_1' and Q_2' are edge disjoint

Dual and facial distance – Meaning

Theorem

G planar graph

$x, y \in V(G)$

- *$d_0^*(x,y)$ is the maximum r such that:*
  G has r edge-disjoint cycles planarly separating x and y.
- *$d_0^*(x,y)$ is the maximum r such that:*
  in any embedding of G there are r nested edge-disjoint cycles planarly separating x and y.
- *$d_0'(x,y)$ is the maximum r such that:*
  G has r vertex-disjoint cycles planarly separating x and y

Get item 2 from item 1 by repeatedly nesting pairs.

Dual and facial distance – Meaning

Corollary

G planar graph
$x, y \in V(G)$
$G - x - y$ max degree Δ

$$\Rightarrow d_0^*(x,y) \leq \lfloor \tfrac{\Delta}{2} \rfloor \cdot d_0'(x,y)$$

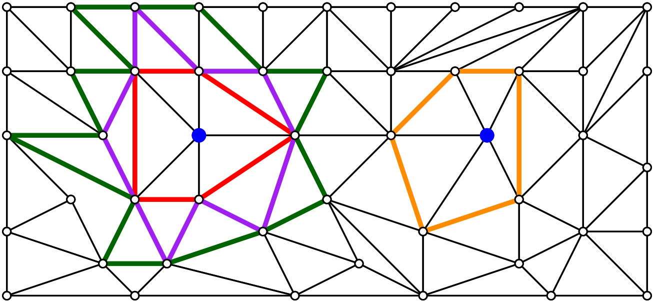
- fix an embedding of G
- take a family of $d_0^*(x,y)$ nested edge-disjoint cycles planarly separating x and y
- select each $\lfloor \frac{\Delta}{2} \rfloor$th cycle

Dual and facial distance – Meaning

Corollary

G planar graph
$x, y \in V(G)$
$G - x - y$ *max degree* Δ

$$\Rightarrow d_0^*(x,y) \leq \lfloor \tfrac{\Delta}{2} \rfloor \cdot d_0'(x,y)$$



Dual and facial distance – Meaning

Corollary

G planar graph
$x, y \in V(G)$
$G - x - y$ max degree Δ

$$\Rightarrow d_0^*(x,y) \le \lfloor \tfrac{\Delta}{2} \rfloor \cdot d_0'(x,y)$$

Corollary

$G - x - y$ max degree 3

$$\Rightarrow d_0^*(x,y) = d_0'(x,y)$$

Outline

Approximating $cr(G + xy)$

- Obvious candidate:
  - embed G such that $d^*(x, y) = d_0^*(x, y)$
  - draw xy on top minimizing crossings
  - crossing number drawing is $d_0^*(x, y)$
- Optimal when G is 3-connected and 3-regular

Approximating $cr(G + xy)$

- Obvious candidate:
  - embed G such that $d^*(x, y) = d_0^*(x, y)$
  - draw xy on top minimizing crossings
  - crossing number drawing is $d_0^*(x, y)$
- Optimal when G is 3-connected and 3-regular
- How good or bad for general graphs?
- analyzed first (and proposed?) by Hliněný & Salazar '06
  - Δ-approximation
- in fact $\lfloor \frac{\Delta}{2} \rfloor$-approximation

Bounding $cr(G+xy)$

Theorem

If G is a planar graph and $x, y \in V(G)$, then

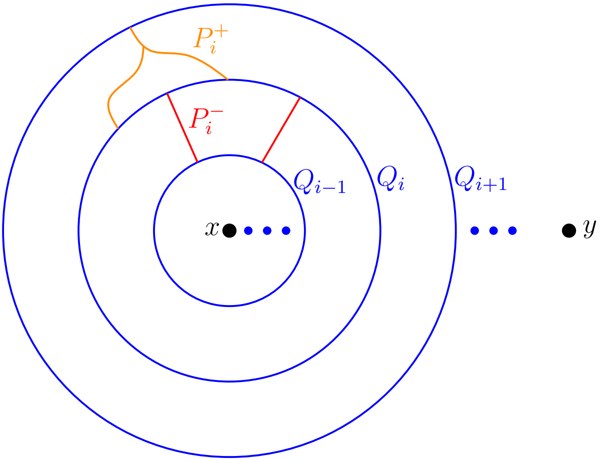
$$d'_0(x,y) \leq cr(G+xy) \leq d^*_0(x,y).$$

- Extends [Riskin'96] since $d'_0 = d^*_0$ for cubic graphs.
- Works also for non-3-connected graphs.
- Right inequality is obvious.
- Let's concentrate on the left inequality.
- Take $r = d'_0(x,y)$.

$d'_0(x,y) \leq cr(G+xy)$ – Nested cycles

- Take r vertex-disjoint cycles $Q_1, \dots Q_r$ that planarly separate x and y
- $Q_0 = x$ and $Q_{r+1} = y$
- Indexed by as nested
- For $1 \leq i \leq r$ bridges $B_x(Q_i)$ and $B_y(Q_i)$ weakly overlap
- Reroute each Q_i such that $B_x(Q_i)$ and $B_x(Q_i)$ overlap
- $Q_1, \dots Q_r$ vertex-disjoint cycles;
  $B_x(Q_i)$ and $B_y(Q_i)$ overlap $(1 \leq i \leq r)$
- P_i^+ paths connecting Q_i to Q_{i+1}
  P_i^- paths connecting Q_i to Q_{i-1}
  $P_i^+ \cup P_i^-$ show overlap of $B_x(Q_i)$ and $B_y(Q_i)$

$d'_0(x, y) \leq cr(G + xy)$ – Nested cycles



$r \leq cr(G + xy)$ – Lower bound

- Consider a drawing of $G + xy$
- Assign to some crossings a label "type i", where $1 \leq i \leq r$
- Assignment is algorithmic
- Argue that for each $1 \leq i \leq r$ there is a crossing of type i

(a) If two edges of the same cycle Q_i cross, we declare such a crossing to be of type i.

(b) If two cycles Q_i and Q_j cross, where $j \neq i$, then they make at least two crossings, and we declare one of them to be a crossing of type i, and another one a crossing of type j.

(c) If the edge xy crosses Q_i, we declare such a crossing to be of type i.

(d) If there are no crossings of type i because of rules (a)–(c), then we consider the set F_i of the edges on the paths $S^1, S^2, \ldots, S^i$ and on the paths $R^i, R^{i+1}, \ldots, R^r$. If an edge in F_i crosses an edge of Q_i, we select one of such crossings and declare it to be of type i.

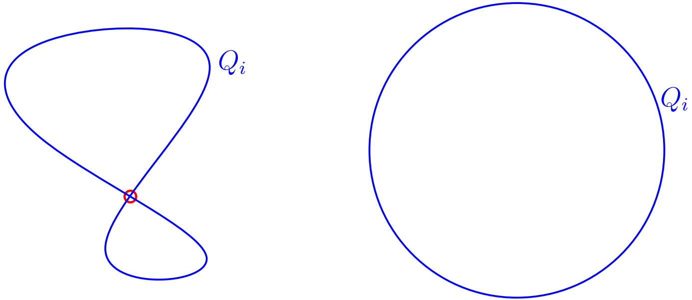
(e) If two edges $e \in E(S^i)$ and $f \in E(R^i)$ cross, we say that the crossing is of type i.

(f) If two edges $e \in E(S^i)$ and $f \in E(Q_{i+1})$ cross and this crossing does not have type $i+1$ assigned by rule (d), we say that this crossing is of type i. Similarly, if two edges $e \in E(R^i)$ and $f \in E(Q_{i-1})$ cross and this crossing does not have type $i-1$ assigned by rule (d), we also say that this crossing is of type i.

(g) Finally, if the cycles Q_{i-1} and Q_{i+1} intersect more than twice, we take one of the intersections that have no type assigned and declare it to be of type i.

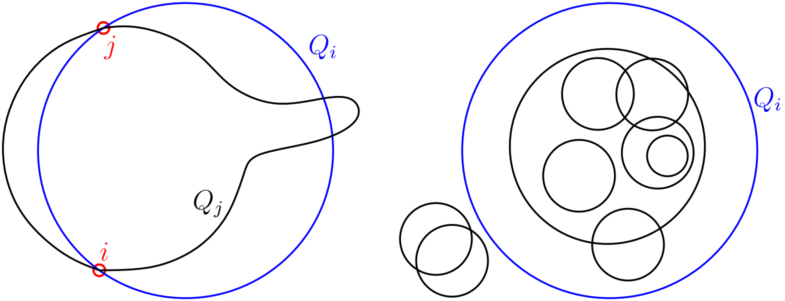
$r \leq cr(G + xy)$ – Assigning types

1. A selfcrossing of Q_i getss type i



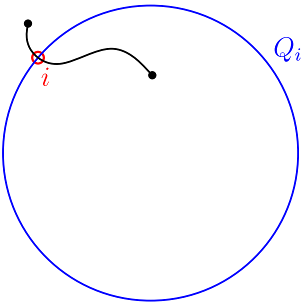
$r \le cr(G+xy)$ – Assigning types

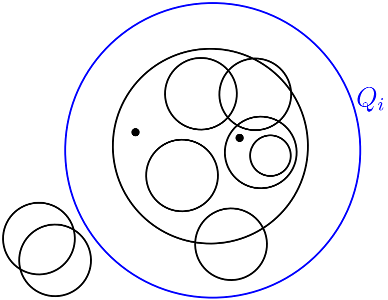
2. If Q_i and Q_j cross, they have ≥ 2 crossings.
   Two such crossings get type i and j



$r \leq cr(G + xy)$ – Assigning types

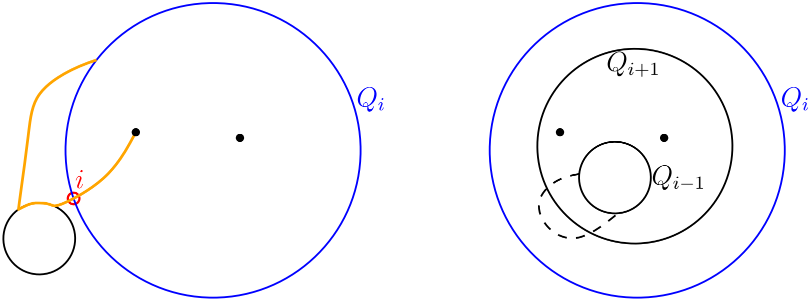
3. If xy crosses Q_i, type i





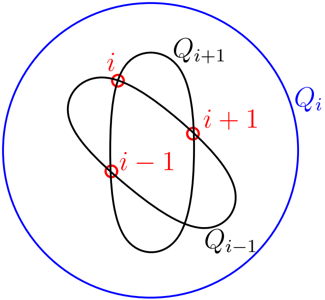
$r \leq cr(G + xy)$ – Assigning types

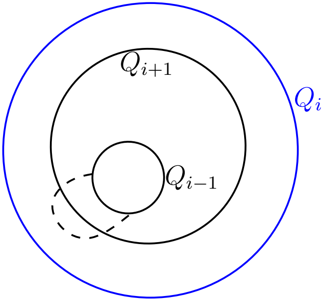
4. If no crossing of type i yet and Q_i crosses $P^-_{\leq i} \cup P^+_{\geq i}$, one such crossing gets type i



$r \leq cr(G + xy)$ – Assigning types

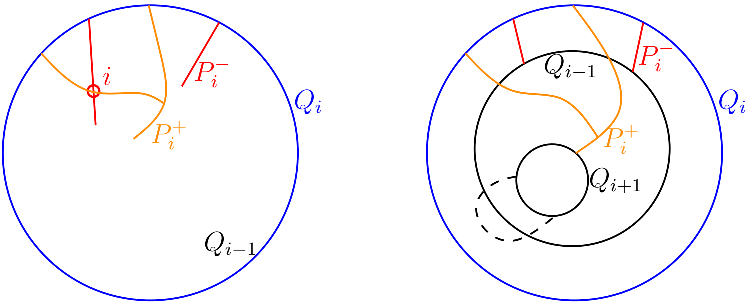
5. If Q_{i-1} and Q_{i+1} cross $\geq$ 4 times, one of the untyped crossings gets type i





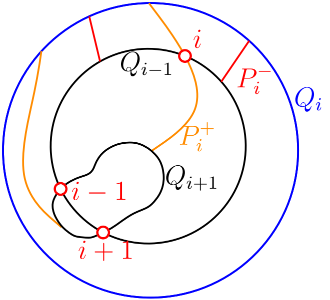
$r \leq cr(G + xy)$ – Assigning types

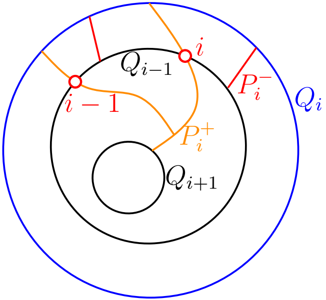
6. A crossing between P_i^- and P_i^+ gets type i



$r \leq cr(G + xy)$ – Assigning types

7. If a crossing between P_i^- and Q_{i+1} has no type $i+1$ assigned yet, gets type i. Similar for P_i^+ and Q_{i-1}.



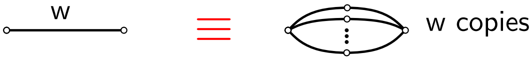


Outline

1. Near-planar graphs
2. Planar separability
3. Dual and facial distances
4. Approximating crossing number
5. Hardness crossing number
6. 1-planarity

Weights

- it is convenient to consider edge-weighted graphs
- positive integer weights
- crossing of edges with weights w and w' give $w \cdot w'$ crossings
- edge of weight $w \equiv w$ parallel subdivided edges



- polynomial vs. non-polynomial weights
- degree = sum of weights of incident edges

Near-planar graphs are hard

Theorem

Computing $cr(G)$ for near-planar graphs is NP-hard.

Near-planar graphs are hard

Theorem

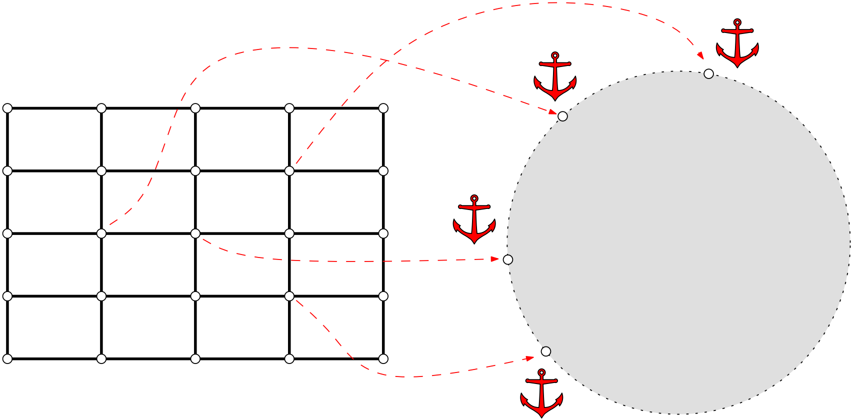
Computing $cr(G)$ for near-planar graphs is NP-hard.

- adding one edge messes up a lot
- easy for *weighted* crossing number
  - polynomial weights would be ok
- new reduction from SAT
  - previous reductions are from Linear Ordering
- new problem: anchored drawings

Tool: anchored drawings

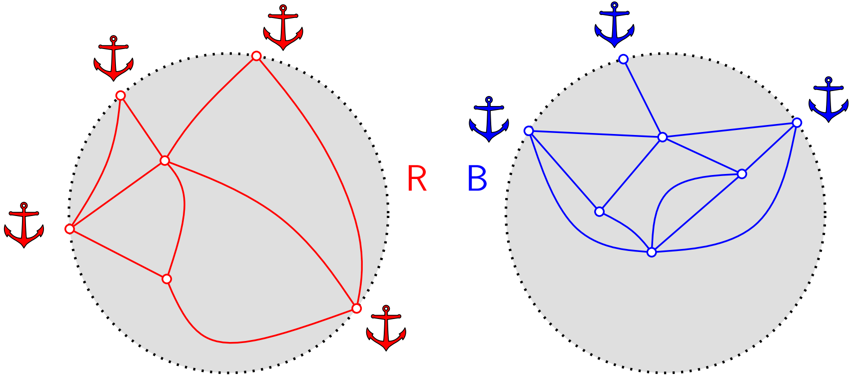
- Ω a disk
- Anchored graph: graph G with assigned placements for a subset $A_G \subseteq V(G)$ of anchors on the boundary of Ω
- Anchored drawing: drawing in Ω extending the placement of A_G
- Anchored embedding: anchored drawing without crossings
- Anchored crossing number: minimize crossings

Tool: anchored drawings



New problem: red-blue anchored drawings

- Ω a disk
- R an anchored embedded red graph in Ω
- B an anchored embedded blue graph in Ω
- anchored crossing number of $R \cup B$



New problem: red-blue anchored drawings

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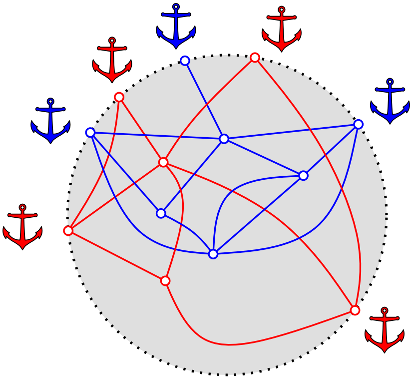


Red-blue anchored drawings

Theorem

It is NP-hard to compute the anchored crossing number of $R \cup B$.

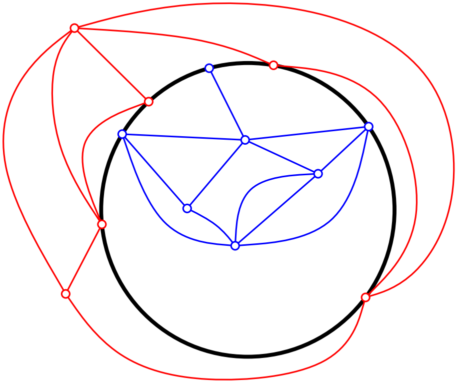
Red-blue anchored drawings

Theorem

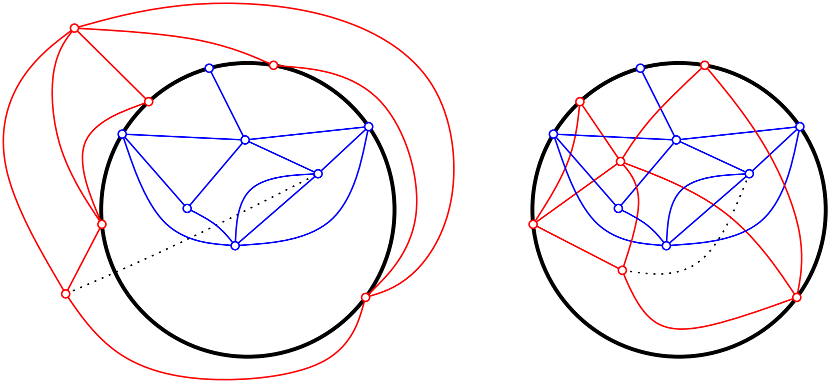
It is NP-hard to compute the anchored crossing number of $R \cup B$.

- also true if R and B disjoint
- also true if restricted to embeddings of R and B
- reduction from SAT

Why red-blue anchored drawings?



Why red-blue anchored drawings?

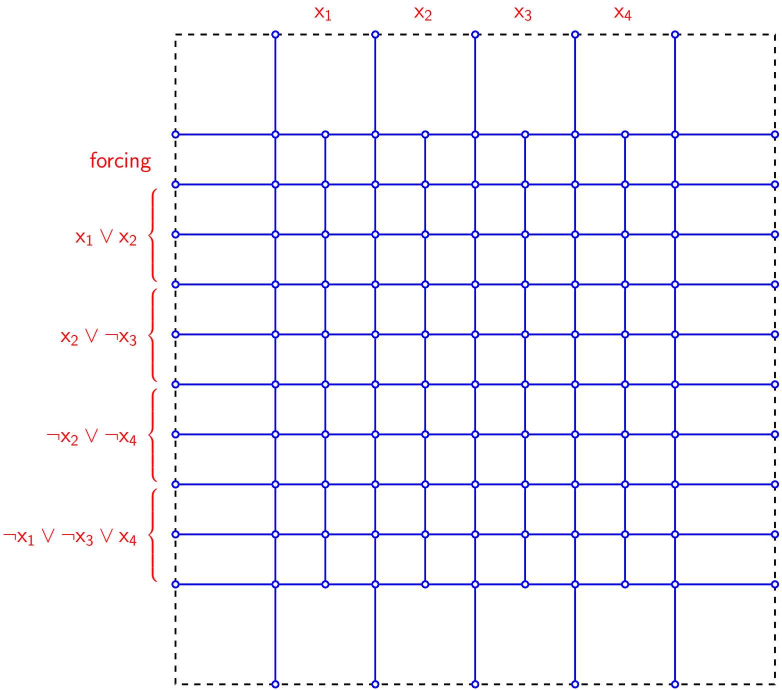


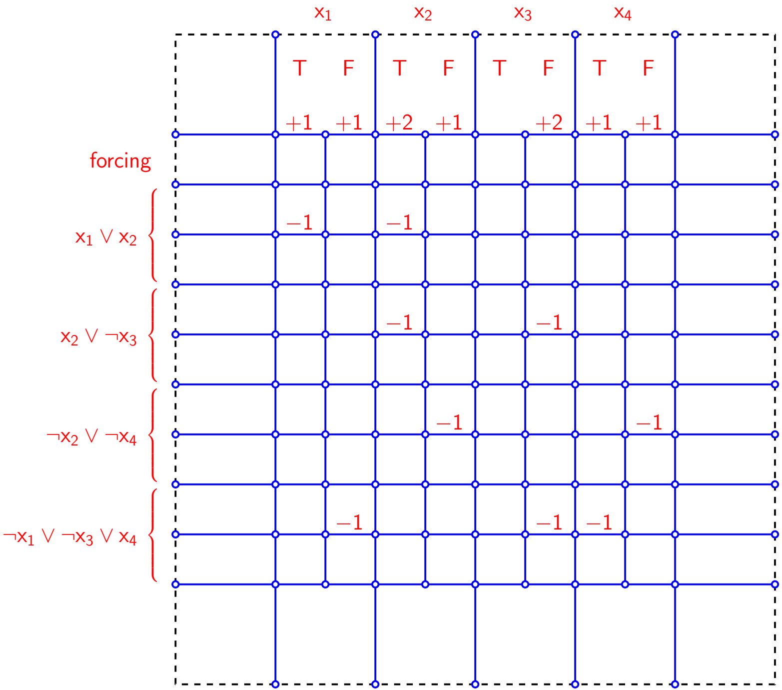
Red-blue anchored drawings

Theorem

It is NP-hard to compute the anchored crossing number of $R \cup B$.

- reduction from SAT
- proof by example
- we will use polynomial weights


x1
x2
x3
x4
forcing
x1 ∨ x2
x2 ∨ ¬x3
¬x2 ∨ ¬x4
¬x1 ∨ ¬x3 ∨ x4


x1
x2
x3
x4
T
F
T
F
T
F
T
F
+1
+1
+2
+1
+2
+1
+1
forcing
x1 ∨ x2
−1
−1
x2 ∨ ¬x3
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¬x2 ∨ ¬x4
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¬x1 ∨ ¬x3 ∨ x4
−1
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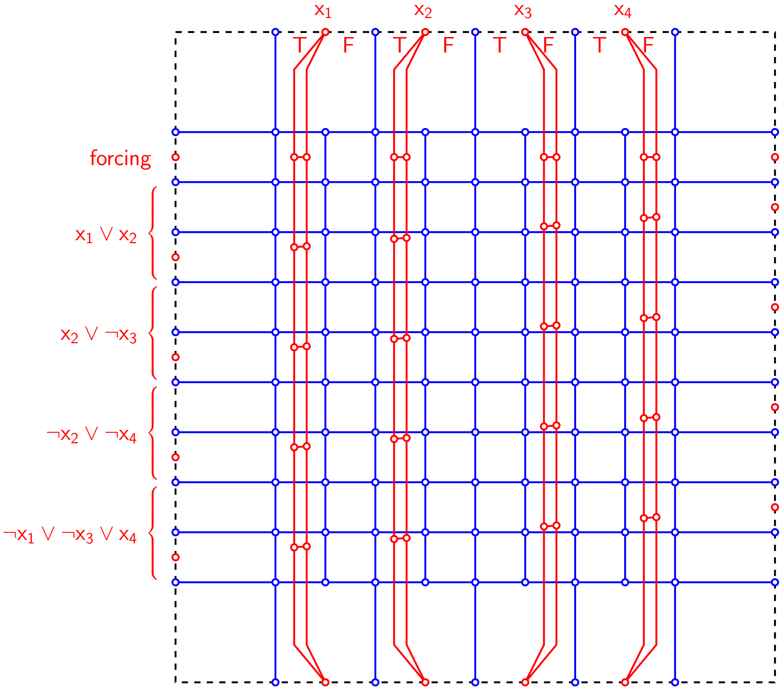
−1



x1
x2
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x4
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F
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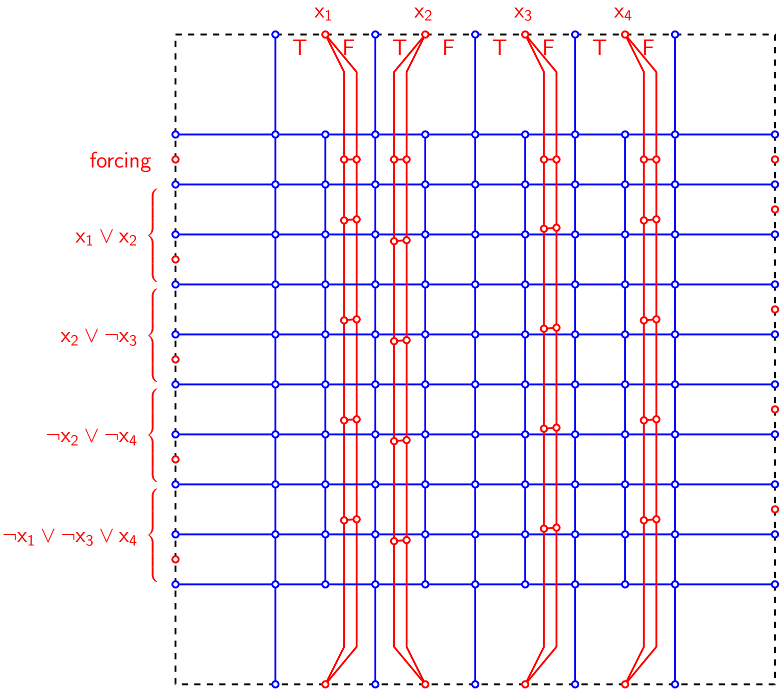
forcing
x1 ∨ x2
x2 ∨ ¬x3
¬x2 ∨ ¬x4
¬x1 ∨ ¬x3 ∨ x4


x_1
x_2
x_3
x_4
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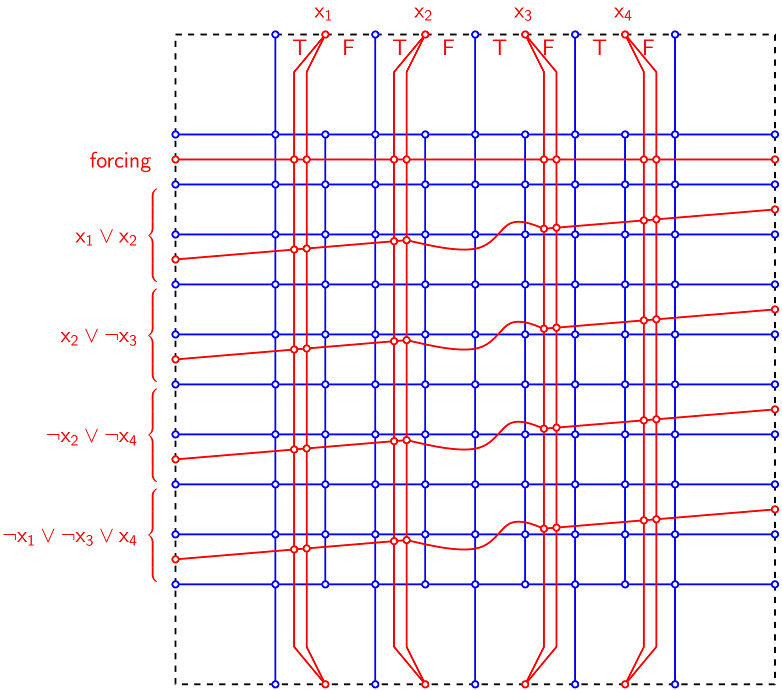
T
F
T
F
T
F
forcing
$x_1 \vee x_2$
$x_2 \vee \neg x_3$
$\neg x_2 \vee \neg x_4$
$\neg x_1 \vee \neg x_3 \vee x_4$



x1
x2
x3
x4
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F
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F
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F
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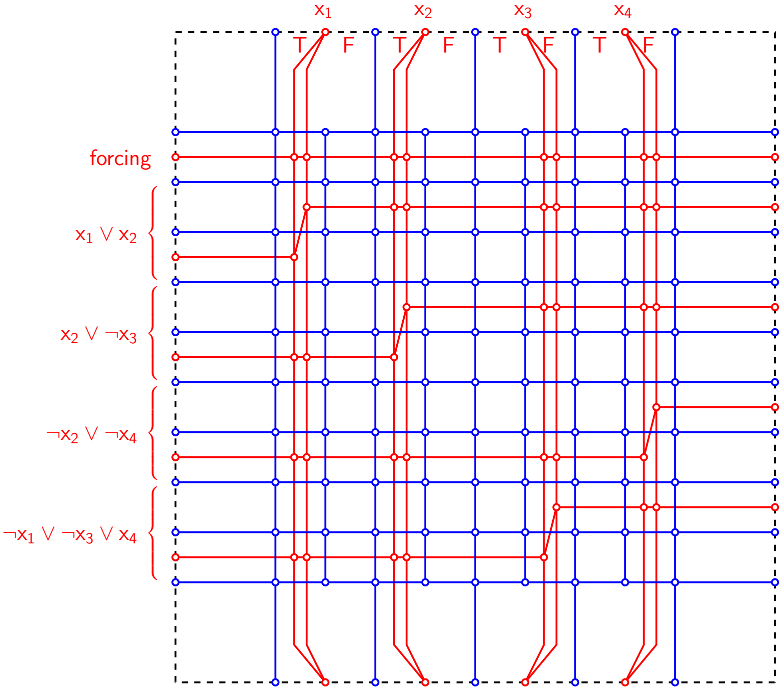
F
forcing
x1 ∨ x2
x2 ∨ ¬x3
¬x2 ∨ ¬x4
¬x1 ∨ ¬x3 ∨ x4



x_1
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x_4
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forcing
$x_1 \vee x_2$
$x_2 \vee \neg x_3$
$\neg x_2 \vee \neg x_4$
$\neg x_1 \vee \neg x_3 \vee x_4$

Low hanging fruit

- Rotation systems
- Cubic graphs
- 3-connected planar with an additional edge
- Planar with rotation systems

Outline

1. Near-planar graphs
2. Planar separability
3. Dual and facial distances
4. Approximating crossing number
5. Hardness crossing number
6. 1-planarity

1-planarity

G is 1-planar if there is a drawing where each edge participates in 0 or 1 crossings.

Theorem

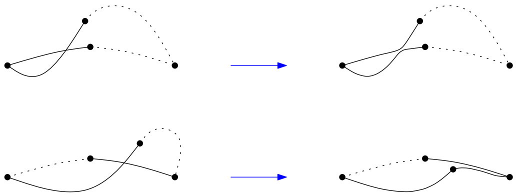
Deciding if a given graph is 1-planar is NP-hard even for near-planar graphs.

- known for general graphs [Korzhik, Mohar '09]
- similar proof technique
- different local structure

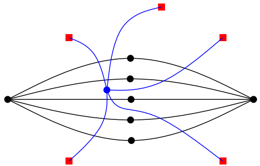
1-planarity – Two tricks

In a 1-planar drawing with fewest crossings

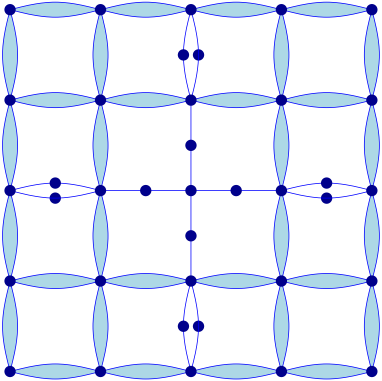
- parallel paths of length 2 do not cross

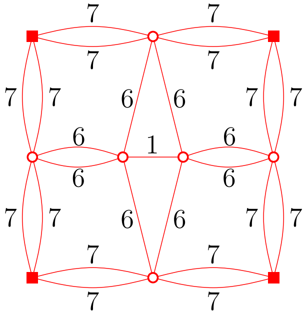


- with some connectivity, no vertex inside faces of parallel edges of length 2

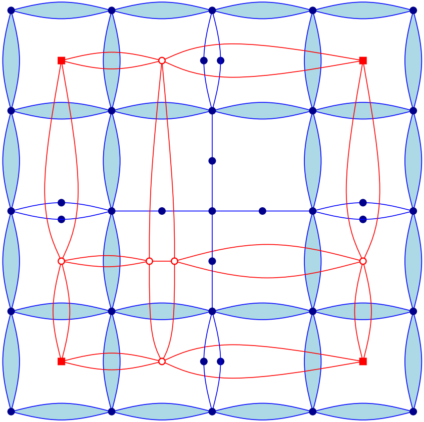


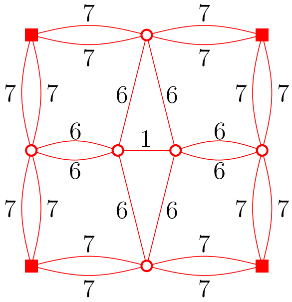
1-planarity – Gadget



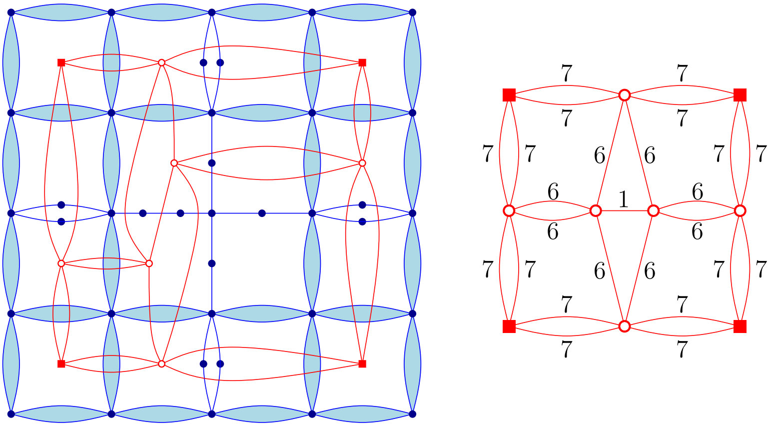


1-planarity – Gadget

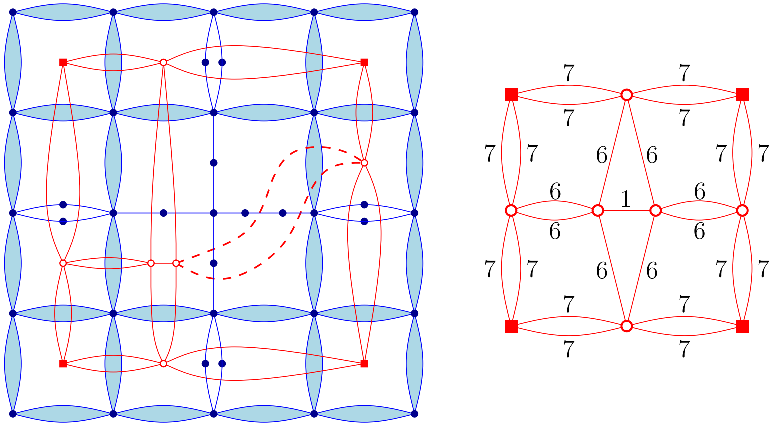




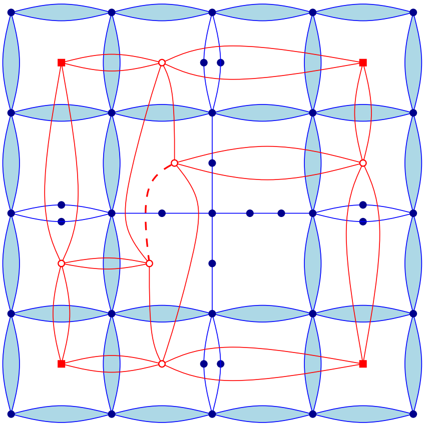
1-planarity – Gadget

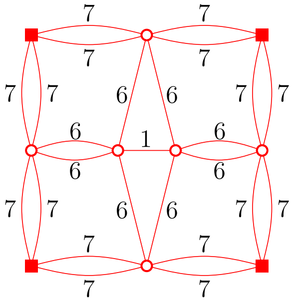


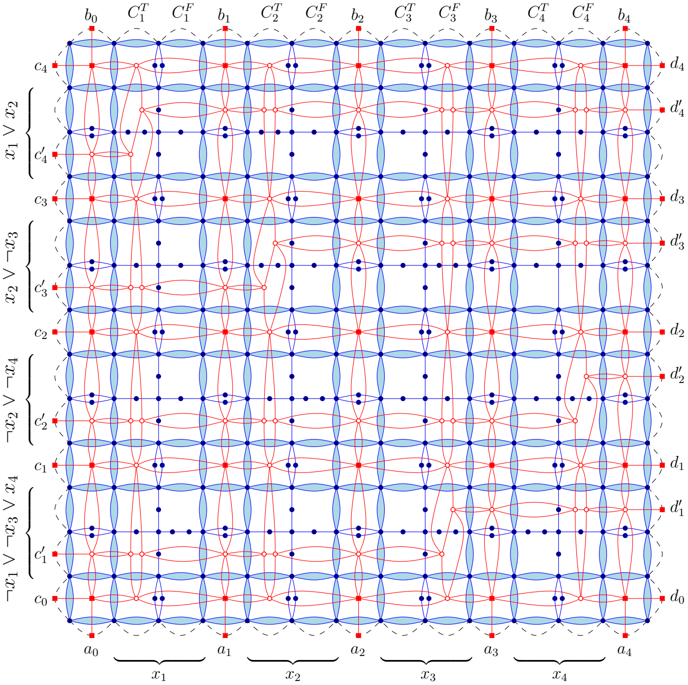
1-planarity – Gadget



1-planarity – Gadget






b_0 C_1^T C_1^F b_1 C_2^T C_2^F b_2 C_3^T C_3^F b_3 C_4^T C_4^F b_4
c_4 c'_4 c_3 c'_3 c_2 c'_2 c_1 c'_1 c_0
d_4 d'_4 d_3 d'_3 d_2 d'_2 d_1 d'_1 d_0
$x_1 \vee x_2$
$x_2 \vee \neg x_3$
$\neg x_2 \vee \neg x_4$
$\neg x_1 \vee \neg x_3 \vee x_4$
a_0 a_1 a_2 a_3 a_4
x_1 x_2 x_3 x_4

Conclusions

- Near-planar graphs are not easy.
- Crossing numbers are hard.
- New problem: anchored drawing in a disk.
- is it hard to compute $cr(G + xy)$ when $\Delta(G) \leq 4$ (via Petr)
- if R and B anchored planar graph, is $cr_a(R \cup B)$ given by a drawing without monochromatic crossings?
- if R and B anchored planar graph with 3 anchors each, can we compute optimal drawing restricted to embeddings in each color?
- crossing number for graphs of bounded treewidth