The rectilinear crossing number of K_n: closing in (or are we?)

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The problem: rectilinear crossing number $\overline{cr}(K_n)$ of K_n

Rectilinear (or _geometric_) drawing of graph G

Every edge is a straight segment

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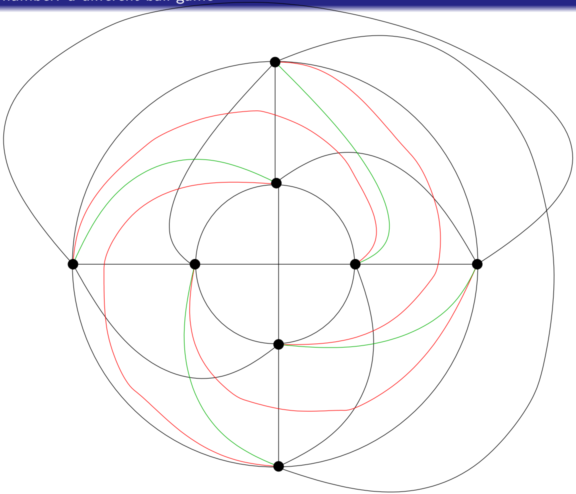
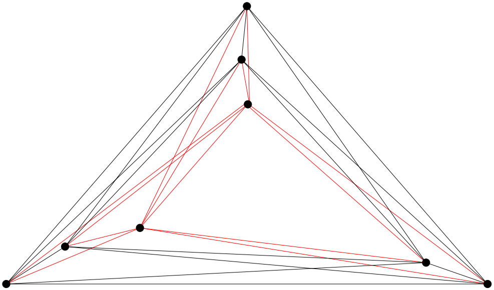
Problem (attributed to Erdös, ca. 1940)

What is $\overline{cr}(K_n)$?



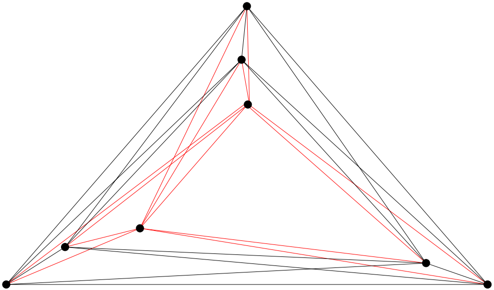
A “usual” drawing of K_8 with 18 crossings.

This is an *optimal* drawing: $\mathrm{cr}(K_8) = 18$.



A *rectilinear* drawing of K_8 with 19 crossings.

This is an *optimal* rectilinear drawing: $\overline{cr}(K_8) = 19$.



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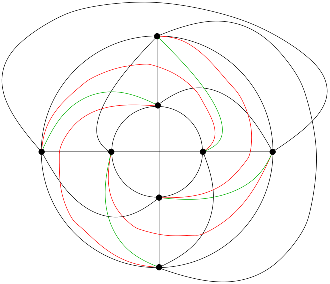
This is an *optimal* rectilinear drawing: $\overline{cr}(K_8) = 19$.

$\mathrm{cr}(K_n)$ and $\overline{cr}(K_n)$ are different, in general

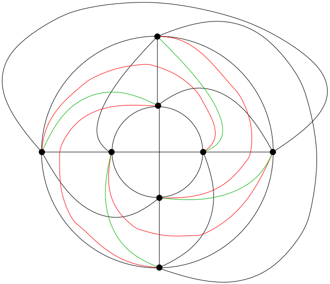
$\overline{cr}(K_n) > \mathrm{cr}(K_n)$, *except* for $n \leq 7$ and $n = 9$.



This type of drawing can be generalized: K_n can be drawn with

$$Z(n) := \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

crossings. Thus $\mathrm{cr}(K_n) \leq Z(n)$.



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Conjecture (Hill, 1959)

$\mathrm{cr}(K_n) = Z(n)$.

Verified for $n \leq 12$. Open for $n > 12$.

We know $\overline{cr}(K_n) > \text{cr}(K_n)$ for $n = 8$ and $n \geq 10$ (more on that, later). But how much bigger? We don't know... we don't know $\overline{cr}(K_n)$ in general. But we know a lot more about $\overline{cr}(K_n)$ than about $\text{cr}(K_n)$:

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- We know $\mathrm{cr}(K_n)$ for $n \leq 12$ (only).
- We know $\overline{cr}(K_n)$ for $n \leq 27$ and $n = 30$.
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- The quotient between the best known lower and upper bounds for $\overline{cr}(K_n)$ is 0.998 (independent of your faith).

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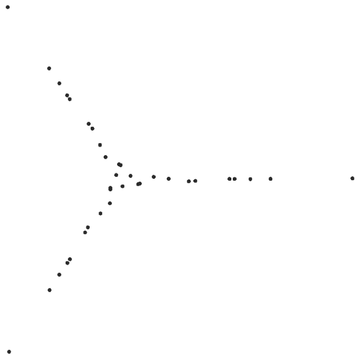
- Close relationship with important parameters in discrete geometry.
- Close relationship (actually, equivalence to) an Erdős-Szekeres type of question (atributed to Erdős).
- Close relationship to Sylvester's Four Point Problem from geometric probability.

(Underlying point set of) rectilinear drawing of K_{51}

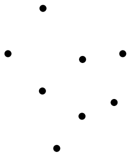


Before we move on: here's a rectilinear drawing of K_{51}. It's relevant (I'll tell you why), and it illustrates well, in a way, how *all* known optimal rectilinear drawings of K_n look like.

Outline

An equivalent (Erdös's actual) question

P a set of n points in the plane in general position
$\square(P) :=$ Number of convex quadrilaterals defined by points in P



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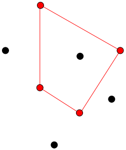

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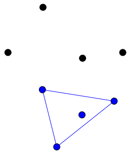

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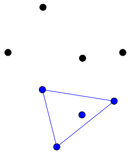

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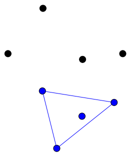

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Question (Erdös, ca. 1940)

What is $\square(n)$?

Observation

$\square(n) = \overline{cr}(K_n)$

The usual reduction

Instead of $\overline{cr}(K_n)$, we focus on

$$q_* := \lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}}$$

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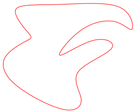
- This is the *proportion* of quadrilaterals that define a crossing, in an optimal rectilinear drawing.
- Only the most optimistic among us (read: $\emptyset$) expects that we'll ever know the *exact* value of $\overline{cr}(K_n)$ for large n. So the asymptotics is a reasonable measure of the quality of our bounds.

Connection to Sylvester's Four Point Problem

Let μ be a probability distribution in the plane (easy case: uniform distribution on your favorite region)



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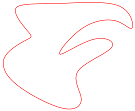

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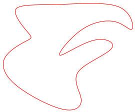

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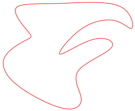

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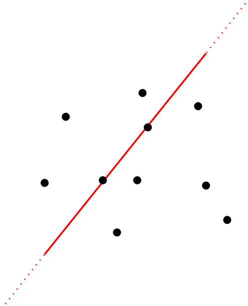
What is *Sylvester's Four Point Constant* $\inf_\mu \Box(\mu)$?

The surprising connection (Scheinerman and Wilf, 1990)

$$\inf_\mu \Box(\mu) = q_* = \lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}}$$

Connections to k-edges and $(\leq k)$-edges

P an n-point set. A *k-edge of P* is a line ℓ that goes through two points p, q or P, and one of the halfplanes defined by ℓ has exactly k points (the other halfplane has $n - k - 2$ points)



A 3-edge (also a 5-edge)

A $(\leq k)$-edge is a j-edge with $j \leq k$ (a 2-edge is a (≤ 2)-edge, also a $(\leq 3) - edge$, also a (≤ 4)-edge, etc.)

Connections to k-edges and $(\leq k)$-edges

P an n-point set. The number of $(\leq k)$-edges in P is $E_{\leq k}(P)$.

Theorem (Lovász, Wagner, Wesztergombi, and Welzl; Ábrego and Fernández-Merchant (2004))

$\overline{cr}(P) = \sum_{k=0}^{\lfloor n/2 \rfloor -2}(n-2k-3)E_{\leq k}(P) +$ smaller order terms

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Lower bounds on $E_{\leq k}(n)$ give lower bounds on $\overline{cr}(K_n)$

$$\overline{cr}(K_n) \geq \sum_{k=0}^{\lfloor n/2 \rfloor - 2} (n-2k-3) E_{\leq k}(n) + \text{ smaller order terms}$$

THIS IS HOW we obtain lower bounds for $\overline{cr}(K_n)$

Halving lines

Another object from discrete geometry

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Maximum number $h(n)$ of halving lines in an n–point set?

Relationship to $\mathrm{cr}(K_n)$

At least for $n \leq 27$ (not known if also for larger n) a point set S minimizes rectilinear crossing number $\leftrightarrow$ S maximizes the number of halving lines

In conclusion, the problem of determining $\mathrm{cr}(K_n)$ has fruitful, interesting, important connections:

- The Erdős-Szekeres problem of determining the minimum number of convex quadrilaterals in an n-point set is equivalent to determining $\mathrm{cr}(K_n)$

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- $\mathrm{cr}(K_n)$ also has close ties to k- and $(\leq k)$-edges — these ties are so close that all the progress in the last 10 years (which is all the substantial progress ever done) depends on this
- Progress on the difficult problem of estimating the number of halving lines has also depended (in the last 10 years) on progress made on $\mathrm{cr}(K_n)$

Progress on $q_* := \lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}}$

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		q_*	$<$	0.3846	(Singer, 1971)
0.2905	$<$	q_*			(Scheinerman and Wilf, 1994)
		q_*	$<$	0.3838	(Brodsky, 2000)
0.3288	$<$	q_*			(Wagner, 2003)
0.37501	$<$	q_*			(Lovasz, Vesztergombi, Wagner, and Welzl, 2004)
		q_*	$<$	0.3807	(Aichholzer and Krasser, 2004)
0.37553	$<$	q_*			(Balogh and S., 2005)
		q_*	$<$	0.38055	(Abrego and Fernandez, 2006)
0.3796	$<$	q_*			(Aichholzer, Orden, Ramos, 2006)
0.37992	$<$	q_*			(Abrego, Fernández, Leaños, and S., 2007)
		q_*	$<$	0.38048	(Abrego, Cetina, Fernández, and S., 2008)

Progress on $q_* := \lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}}$

Current best

$$0.37992 < q_* < 0.38048$$

$$\frac{0.37992}{0.38048} \approx 0.998$$

Same story for **EXACT** results: stuck, then progress

Exact results

- We know the **exact** value of $\overline{cr}(K_n)$ for $n \leq 27$ (Ábrego, Fernández, Leaños, and S., 2007).
- $\overline{cr}(K_{30})$ is also known (Cetina, Hernández-Vélez, Leaños, 2010)

What happened?

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However, in 2010, $\overline{cr}(K_{30}) = 9726$ was proved...in 10 pages (!).

What happened?

In 2003/2004, Ábrego-Fernández-Merchant, and, independently, Lovász-Vesztergombi-Wagner-Welzl discovered (and exploited) the relationship between $\overline{cr}(K_n)$ and $(\leq k)$-edges:

Theorem (Lovász, Wagner, Wesztergombi, and Welzl; Ábrego and Fernández-Merchant (2004))

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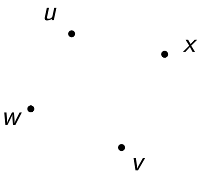
Lower bounds on $E_{\leq k}(n)$ give lower bounds on $\overline{cr}(K_n)$

$$\overline{cr}(K_n) \geq \sum_{k=0}^{\lfloor n/2 \rfloor - 2} (n-2k-3) E_{\leq k}(n) + \text{ smaller order terms}$$

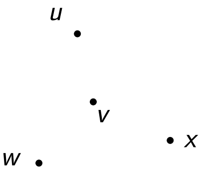
The crucial observation

Points u, v, w, x. How many ordered 4-tuples on these points there exist, such that the line spanning u and v *separates* w and x?

- 4 ways, if u, v, w, x form a convex quadrilateral
- 6 ways, if u, v, w, x **don't** form a convex quadrilateral



Convex position



Nonconvex position

This observation + some easy counting...

Let $\square(P)$ denote the number of convex quadrilaterals of P.

Let $e_j(P)$ denote the number of j-edges of P.

$$\square(P) = \sum_{j < \frac{n-2}{2}} e_j(P) \left(\frac{n-2}{2} - j \right)^2 - \frac{3}{4} \binom{n}{3}$$

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The (huge) consequence

Want to count crossings? Count $(\leq k)$-edges.

$$\square(P) = \sum_{k=0}^{\lfloor n/2 \rfloor - 2} (n-2k-3) E_{\leq k}(P) + \text{ smaller order terms}$$

- $\square(n) :=$ minimum $\square(P)$ over all n-point sets P
- $E_{\leq k}(n) :=$ minimum ${}_{\leq k}(P)$ over all n-point sets P

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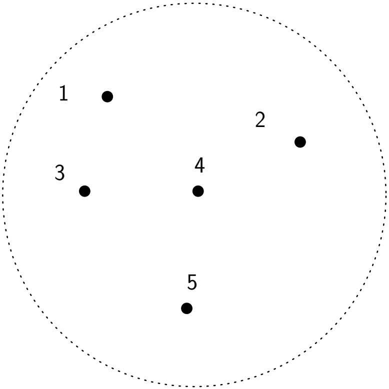
Recall: $E_{\leq k}(n) :=$ minimum $E_{\leq k}(P)$ over all n-point sets P

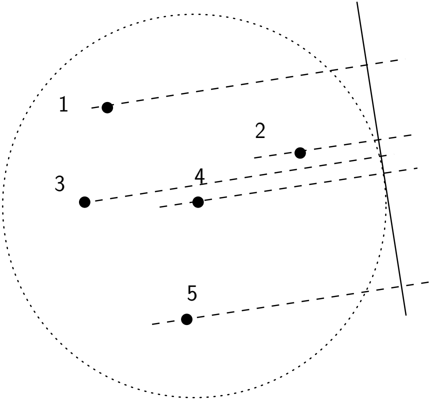
Lower bounds on $E_{\leq k}(n) \implies$ lower bounds on $\overline{cr}(K_n)$

But... how do we obtain lower bounds on $E_{\leq k}(n)$?

Circular sequences

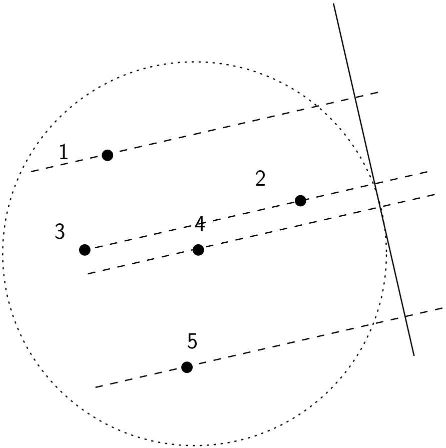


Circular sequences



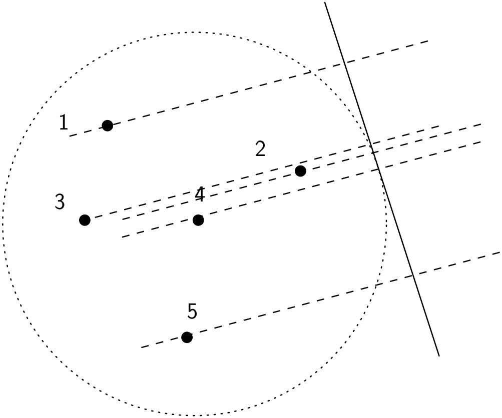
1 2 3 4 5

Circular sequences



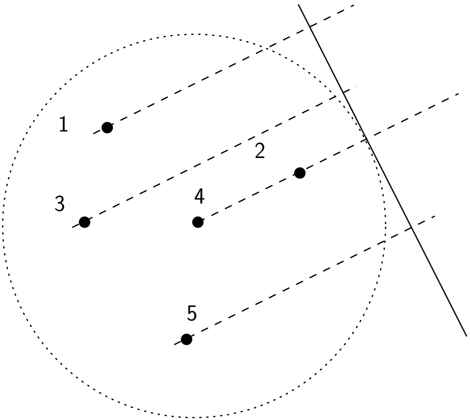
1 2 3 4 5

Circular sequences



1	2	3	4	5
1	3	2	4	5

Circular sequences



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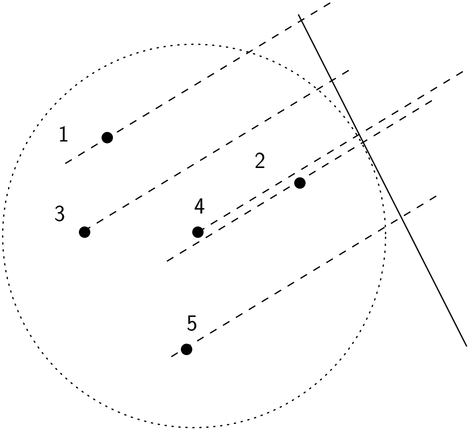
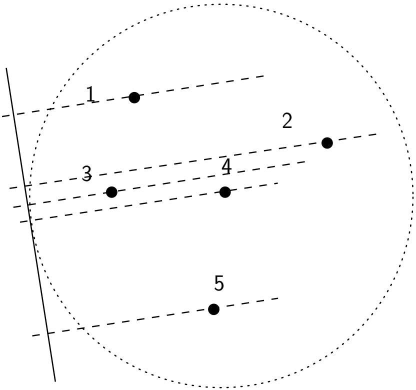
Circular sequences



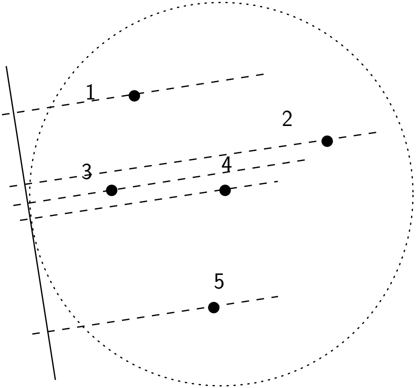
1	2	3	4	5
1	3	2	4	5
1	3	4	2	5

Circular sequences



1	2	3	4	5
1	3	2	4	5
1	3	4	2	5
1	3	4	5	2
3	1	4	5	2
3	1	5	4	2
3	5	1	4	2
5	3	1	4	2
5	3	4	1	2
5	3	4	2	1
5	4	3	2	1

Circular sequences



1	2	3	4	5
1	3	2	4	5
1	3	4	2	5
1	3	4	5	2
3	1	4	5	2
3	1	5	4	2
3	5	1	4	2
5	3	1	4	2
5	3	4	1	2
5	3	4	2	1
5	4	3	2	1

This sequence of permutations is the **circular sequence** $\Pi(P)$ of P

Circular sequences

The circular sequence $\Pi(P)$ of P encodes valuable geometrical information.

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In particular, $(\leq k)$-edges are very easy to identify...

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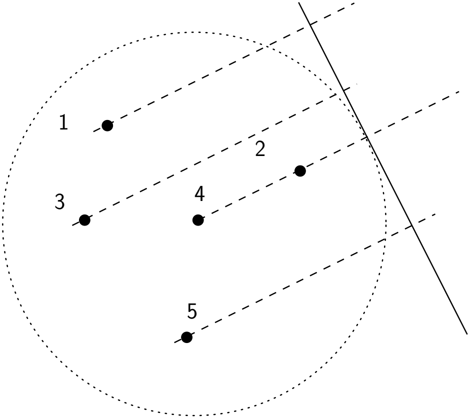
1	3	4	5	2
3	1	4	5	2
3	1	5	4	2
3	5	1	4	2
5	3	1	4	2
5	3	4	1	2
5	3	4	2	1
5	4	3	2	1

Circular sequences



1	2	3	4	5
1	3	2	4	5
1	3	4	2	5
1	3	4	5	2
3	1	4	5	2
3	1	5	4	2
3	5	1	4	2
5	3	1	4	2
5	3	4	1	2
5	3	4	2	1
5	4	3	2	1

This transposition identifies a 2-edge (a 1-edge as well)

Circular sequences

The circular sequence is a sequence of transpositions. Each transposition is a k-edge for some k — it suffices to see how many points the transposing elements have to their left (or right).

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5	3	1	4	2
5	3	4	1	2
5	3	4	2	1
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Circular sequences: **the** tool to bound the number of $(\leq k)$-edges.

$E_{\leq k}(n) :=$ minimum $E_{\leq k}(P)$ over all n-point sets P

$$\overline{cr}(K_n) = \Box(n) \geq \sum_{k=0}^{\lfloor n/2 \rfloor - 2} (n-2k-3)E_{\leq k}(n) + \text{ smaller order terms}$$

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In order to (lower) bound $E_{\leq k}(n)$...

It suffices to check, over **all** circular sequences on n elements, which one has the smallest number of transpositions involving the leftmost or rightmost k columns

Of course, a lot easier to say than to do, but still...

1	2	3	4	5	6	7	8	9
1	2	4	3	5	6	7	8	9
1	4	2	3	5	6	7	8	9
4	1	2	3	5	6	7	8	9
4	1	2	5	3	6	7	8	9
4	1	5	2	3	6	7	8	9
4	5	1	2	3	6	7	8	9
4	5	1	2	6	3	7	8	9
4	5	1	6	2	3	7	8	9
4	5	6	1	2	3	7	8	9
4	5	6	2	1	3	7	8	9
4	5	6	2	3	1	7	8	9
4	5	6	3	2	1	7	8	9
4	5	6	3	2	7	1	8	9
4	5	6	3	2	7	8	1	9
4	5	6	3	2	7	8	9	1
4	5	6	3	7	2	8	9	1
4	5	6	3	7	8	2	9	1
4	5	6	3	7	8	9	2	1

4	5	6	7	3	8	9	2	1
4	5	6	7	8	3	9	2	1
4	5	6	7	8	9	3	2	1
4	5	6	7	9	8	3	2	1
4	5	6	9	7	8	3	2	1
4	5	6	9	8	7	3	2	1
4	5	9	6	8	7	3	2	1
4	9	5	6	8	7	3	2	1
9	4	5	6	8	7	3	2	1
9	4	5	8	6	7	3	2	1
9	4	8	5	6	7	3	2	1
9	8	4	5	6	7	3	2	1
9	8	4	5	7	6	3	2	1
9	8	4	7	5	6	3	2	1
9	8	7	4	5	6	3	2	1
9	8	7	4	6	5	3	2	1
9	8	7	6	4	5	3	2	1
9	8	7	6	5	4	3	2	1

- 3 0-edges
- 6 1-edges
- 9 2-edges
- 18 3-edges

There's a clear pattern: 3, 6, 9, ... This is **not** a coincidence.

In the crossing-minimal “drawing” I just showed you, there are:

- 3 0-edges
- 6 1-edges
- 9 2-edges
- 18 3-edges

(that’s all; with 9 points, a 4-edge is also a 3-edge)

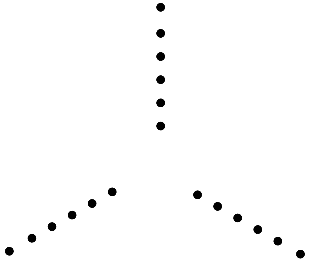
There is a clear pattern, $3, 6, 9 \ldots$ This was noticed by Ábrego-Fernández and Lovász et al.:

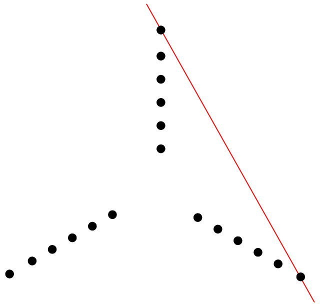
Bound for ($\leq k$)–edges (using circular sequences)

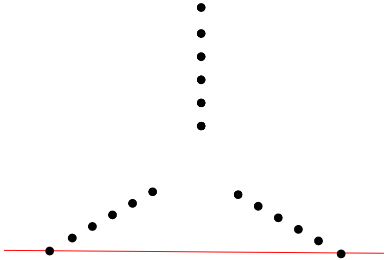
$$E_{\leq k}(n) \geq 3\binom{k+1}{2}$$

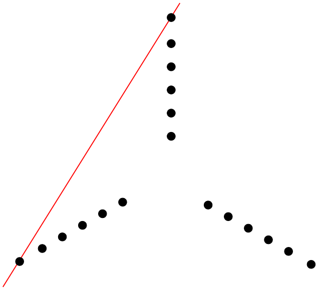
This bound is actually tight for $k \leq n/3 - 1$.

A crossing-minimal "drawing" of K_9

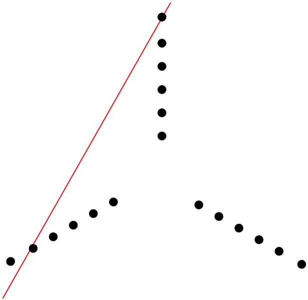


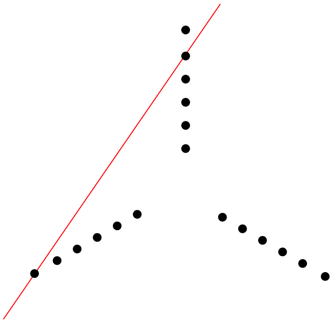


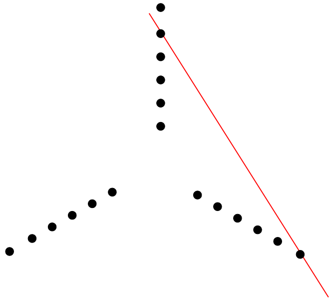


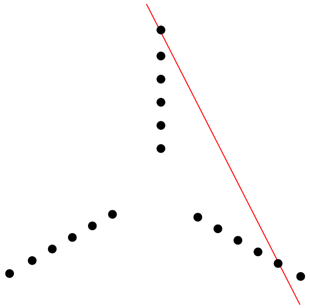


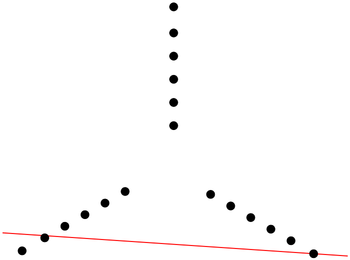
There are 3 0-edges

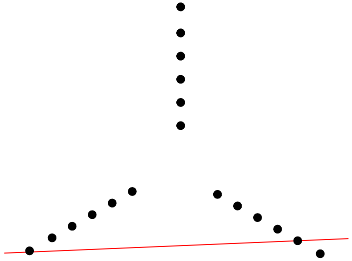












There are 6 1-edges (and so 9 (≤ 1)-edges)

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So this bound is actually <u>tight</u> for $k \leq n/3 - 1$.

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It is **not** tight for $k > n/3 - 1 \ldots$ room for improvement!

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Using these two ingredients, by an elementary calculation. . .

$$q_* := \lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \geq 0.375$$

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Actually, Lovász et al. went a little further:

Bound obtained by Lovász et al.

$$q_* := \lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} > 0.37501$$

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$$\mathrm{cr}(K_n) \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

Thus

$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}} \leq \lim_{n\to\infty} \frac{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor}{\binom{n}{4}} = 0.375$$

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This explains why Lovász would bother with 0.00001...

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} > \lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$$

Current best lower bound

Bound for ($\leq k$)–edges (using circular sequences)
Ábrego, Cetina, Fernández-Merchant, Leaños, S., 2008

$$E_{\leq k}(n) \geq 3\binom{k+1}{2} + 3\binom{n-k/3+1}{2} + \text{ugly stuff}$$

- "ugly stuff" only applies to $k > 4n/9$;
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- "ugly stuff" only applies to $k > 4n/9$;
- this inequality is sharp for $k \leq 4n/9$

Using this, by an elementary calculation...

$$q_* := \lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} > 0.37992$$

Application to exact results

Bound for $(\leq k)$–edges

$$E_{\leq k}(n) \geq 3\binom{k+1}{2} + 3\binom{n-k/3+1}{2} + \text{ugly stuff}$$

Using this, and the equation that relates $\overline{cr}(K_n)$ to $E_{\leq k}(n)$, very easy calculations give:

Exact results

- We know the **exact** value of $\overline{cr}(K_n)$ for $n \leq 27$ (Ábrego, Fernández, Leaños, and S., 2007).
- $\overline{cr}(K_{30})$ is also known (Cetina, Hernández-Vélez, Leaños, 2010)

Minimizing ($\leq k$)-edges **and** crossing number

SO FAR

($n \leq 27$, the values for which we know the exact value of $\overline{cr}(K_n)$):

n–point set S minimizes $\overline{cr}(K_n)$
if and only if
for $k = 1, 2, \ldots, n/2 - 1$, S minimizes $E_{\leq k}(n)$
and, consequently, if and only if
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But we have evidence that indicates this won't be the case for larger values of n. . .

Remark on the upper bounds. . . we just **don't** have any "natural" geometric drawings of K_n!

For all popular families of graphs, we easily come up with natural drawings with (apparently) few crossings:

- Automatically, **any** drawing gives an upper bound
- Eventually, a drawing survives the test of time – and you have a conjecture for the crossing number of your graph

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At a somewhat philosophical level, the lack of an aim makes things even harder for the lower bounds side — normally the **only** side we need to work on

What we do to get the best upper bounds available

The paradigm (Brodsky et al.; Aichholzer et al.)

- Start with some drawing of K_p
- Substitute each point with a cluster of points
- Design a crossing–friendly layout for each cluster

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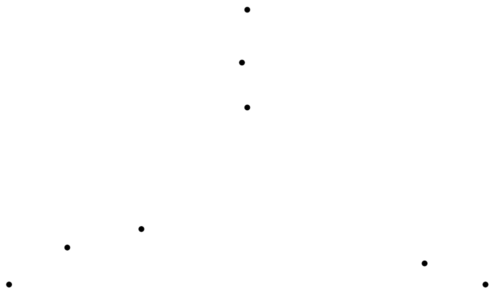
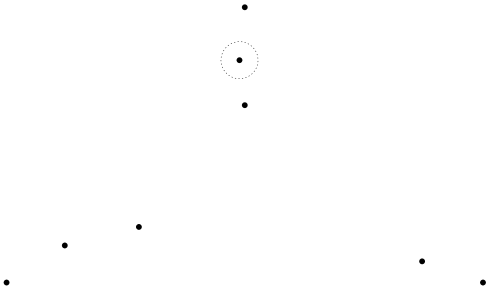
The gory details...

Start with (NOT OPTIMAL) drawing of K_{51}, get drawing of K_{505}, then iteratively drawings of $K_{2^N \cdot 505}$.



Drawing of K_n (some n)



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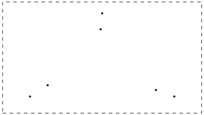
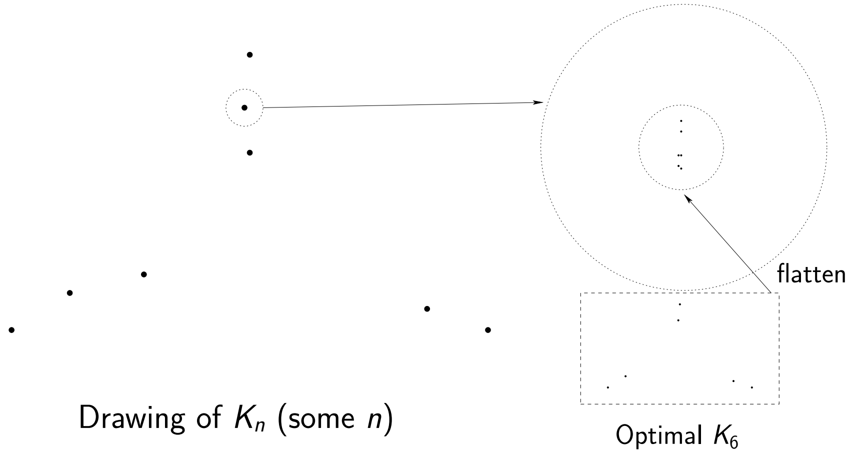
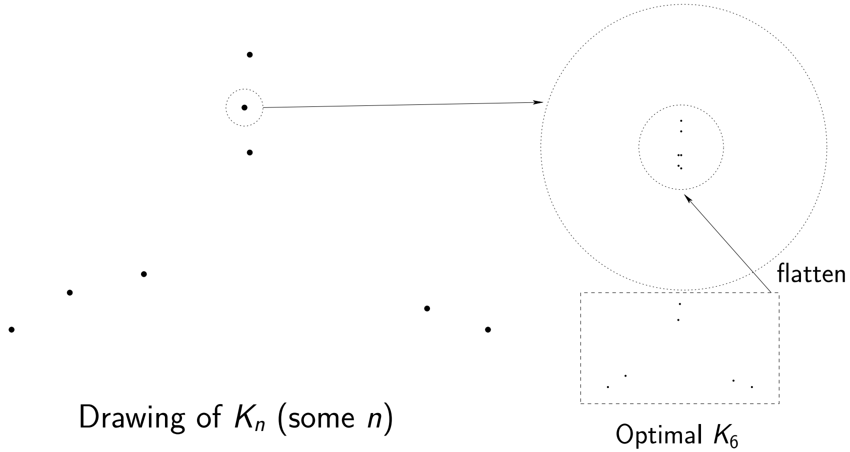
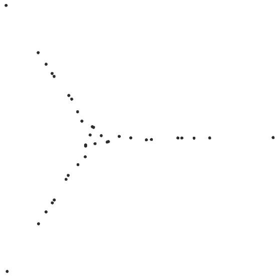


Optimal K_6

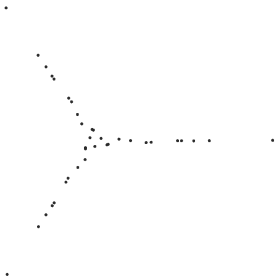

flatten
Drawing of K_n (some n)
Optimal K_6



If you say “hey, this is voodoo!”, I reply: “You’re missing the point! It’s the **most successful voodoo** around!”

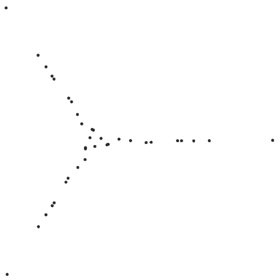


Underlying point set of a **non optimal** rectilinear drawing of K_{51}



Underlying point set of a **non optimal** rectilinear drawing of K_{51}

This is the “base” drawing that has given us the best results: substitute each point by a K_r (for different values of r), to get a drawing of K_{505}.



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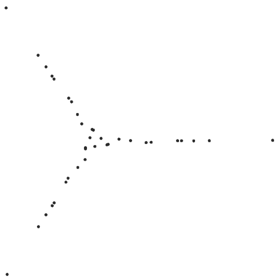
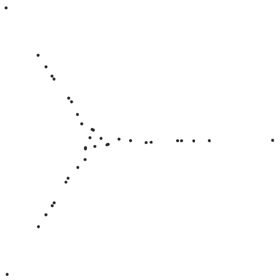


Underlying point set of a **non optimal** rectilinear drawing of K_{51}

This is the "base" drawing that has given us the best results: substitute each point by a K_r (for different values of r), to get a drawing of K_{505}. Then duplicate each point over and over, and get a drawing of $K_{2^N \cdot 505}$. Such drawings are the best (!) drawings known of K_n for n large.



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Yes, when I said "voodoo" I wasn't being modest. . .

We just don't know how to produce good candidates for optimal rectilinear drawings of K_n

More general than rectilinear drawings!

Circular sequences (Goodman and Pollack, 1980)

Encode all the geometrical information of an n–point set in a sequence of $\binom{n}{2}$ permutations on n symbols

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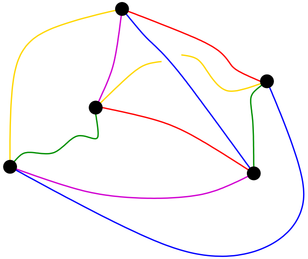
Every point set yields a circular sequence, but **not every circular sequence comes from a point set**

Pseudolinear drawings

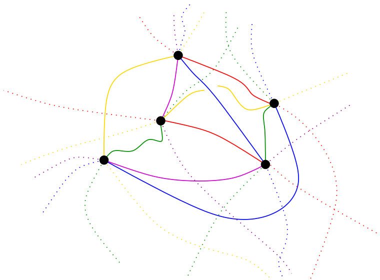
A non–rectilinear drawing of K_n:



Still, not an "arbitrary" drawing...

Pseudolinear drawings

We may extend each edge to a (pseudo)line, so that the result is an *arrangement of pseudolines*: every two of them cross each other exactly once:



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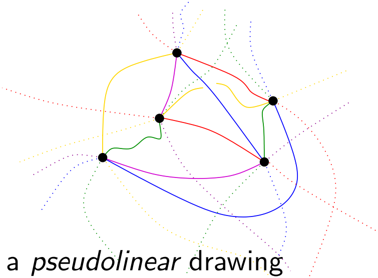

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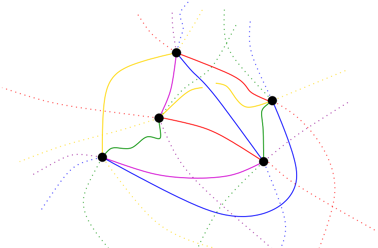

Such a drawing is a *pseudolinear* drawing

Correspond bijectively with circular sequences

Every circular sequence corresponds to a pseudolinear drawing (Goodman and Pollack, 1980)

Rectilinear vs. Pseudolinear

Key observation

Every rectilinear drawing is also pseudolinear, but not the other way around

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And (make it **BUT**) . . . (the bad news)

The **upper** bounds we prove using circular sequences are **not** upper bounds for rectilinear drawings

Conclusions

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Thank you for your attention!